$\boxed{\text{DIFFRACTION}}$

Reading: Gasvik CH. 4

MAXWELLS EQUATIONS

• Govern behavior of electromagnetic fields

$$\nabla \cdot \vec{E} = \frac{\rho_f}{\epsilon} \quad \text{GAUSS' LAW} \quad \left(\int q_i = \int_s \vec{E} \cdot dA\right)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{FARADAY'S LAW OF INDUCTION}$$

$$\nabla \cdot \vec{B} = 0 \quad \text{NO MAGNETIC MONOPOLES}$$

$$\nabla \times \vec{B} = \mu \sigma \vec{E} + \mu \epsilon \frac{\partial \vec{E}}{\partial t} \quad \text{MAGNETIC INDUCTION (Displacement Current)}$$

ρ_f = free charge density, $\vec{J}_f'$ = free charge current, μ = magnetic permeability, ϵ = electric permittivity

σ = conductivity $\quad \vec{J}_f = \sigma \vec{E} + \vec{J}_f'$

- ELIMINATE B:

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\nabla \cdot (\underbrace{\cancel{\nabla \cdot \vec{E}}}_{0}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{B}) \qquad \text{Assume } \rho_f = 0$$

$$-\nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\cancel{\mu \sigma \vec{E}} + \mu \epsilon \frac{\partial \vec{E}}{\partial t}\right) \qquad \text{Assume } \sigma = 0 \text{ (dielectric)}$$

$$\boxed{\nabla^2 \vec{E} - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t} = 0} \quad \text{3D WAVE EQUATION FOR ELECTRIC FIELD}$$

$$\frac{\partial^2 \psi}{\partial z} - \mu \epsilon \frac{\partial^2 \psi}{\partial t} = 0 \quad \text{1D WAVE EQUATION} \quad \text{– HYPERBOLIC 2ND ORDER PDE}$$

- LET: $v = \frac{1}{\sqrt{\mu \epsilon}} = \frac{c}{n}$ ← speed of light $3e^8 \left[\frac{m}{sec}\right]$, n ← index of refraction; v = wave velocity $\rightarrow \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}$
- SEPARATION OF VARIABLES $\quad \psi(z,t) = Z(z)\,T(t)$

$$\frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = \frac{1}{v^2 T} \frac{\partial^2 T}{\partial t^2} = \text{CONST} = -k^2$$

- MOMENTUM $\quad k = \frac{\omega N}{c} = \frac{2\pi}{\lambda}$, $\quad N = n + ib$, $\quad \lambda \nu = v$

  k ← Propagation constant (momentum); N = COMPLEX OPTICAL INDEX; ν → FREQUENCY; λ → WAVELENGTH

PLANE WAVES

- WAVE EQUATION HAS 2 TYPES OF SOLUTION

$$Z_k(z) = \alpha_k e^{ikz} + \beta_k e^{-ikz}$$
$$T_k(z) = \gamma_k e^{i\omega t} + \delta_k e^{-i\omega t}$$

SUPERPOSITION

$$\psi(z,t) = \sum_k \left[\alpha_k \delta_k e^{i(kz-\omega t)} + \beta_k \gamma_k e^{-i(kz-\omega t)}\right] + \sum_k \left[\alpha_k \gamma_k e^{i(kz+\omega t)} + \beta_k \delta_k e^{-i(kz+\omega t)}\right]$$

• Pick only $\omega > 0$, $k \pm$ determines direction

$$\Psi(z,t) = \Psi_0 e^{i(kz-\omega t)}$$
CONSTANT

IDENTITIES

$$e^{iu} = \cos u + i \sin u \qquad \cos z = \frac{e^{iz}+e^{-iz}}{2}$$

$$e^{-iu} = \cos u - i \sin u \qquad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

$$\Psi = (\Psi_{0R} + i\Psi_{0I})\left[\cos(kz-\omega t) + i \sin(kz-\omega t)\right]$$

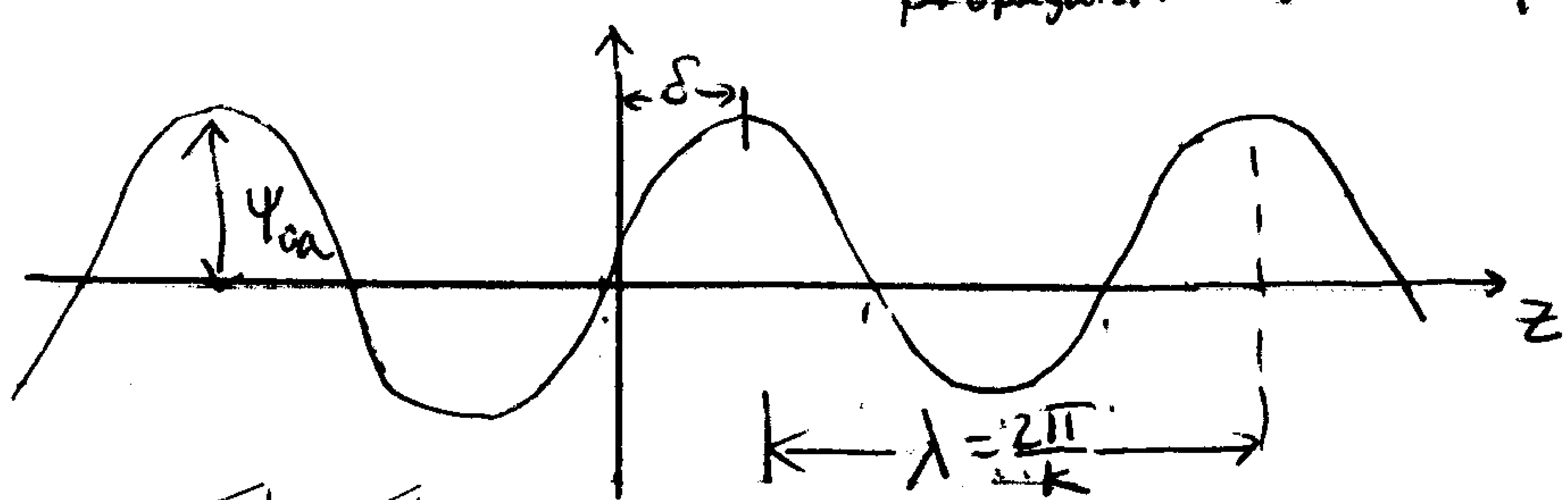
$$\Psi_{physical} = Re(\Psi) = \boxed{\Psi_{0R}\cos(kz-\omega t) - \Psi_{0I}\sin(kz-\omega t)}$$

CONSIDER CONSTANT $\Psi_0 = \Psi_{0a} e^{i\delta} \rightarrow \Psi_{0R} = \Psi_{0a}\cos\delta,\ \Psi_{0I} = \Psi_{0a}\sin\delta$

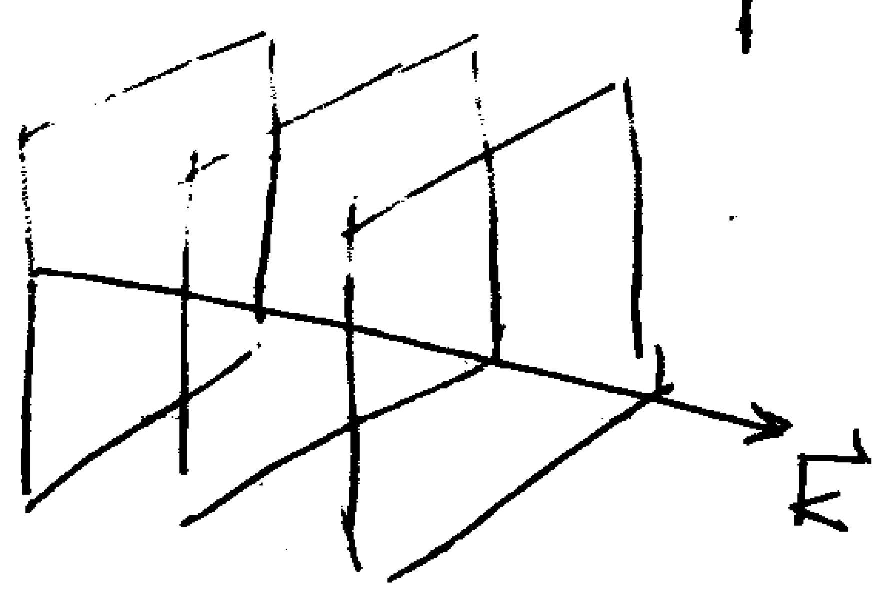
$$\Psi_{phys} = \Psi_{0a}\cos(kz - \omega t + \delta)$$
phase factor

spatial propagation

angular frequency

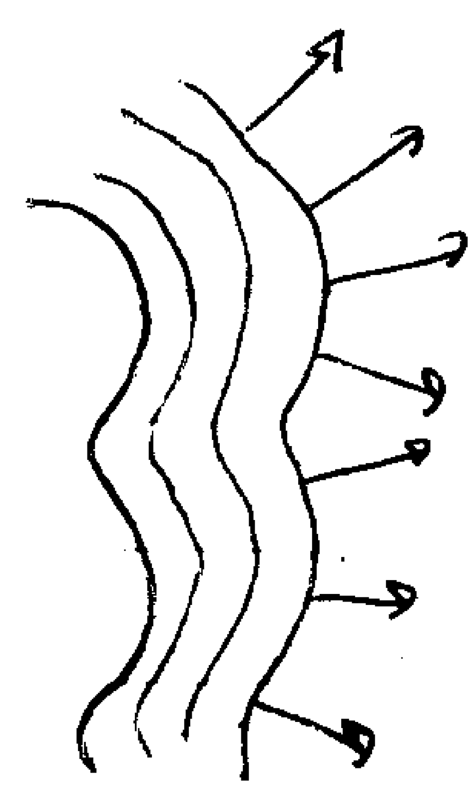


SPATIAL COMPONENT

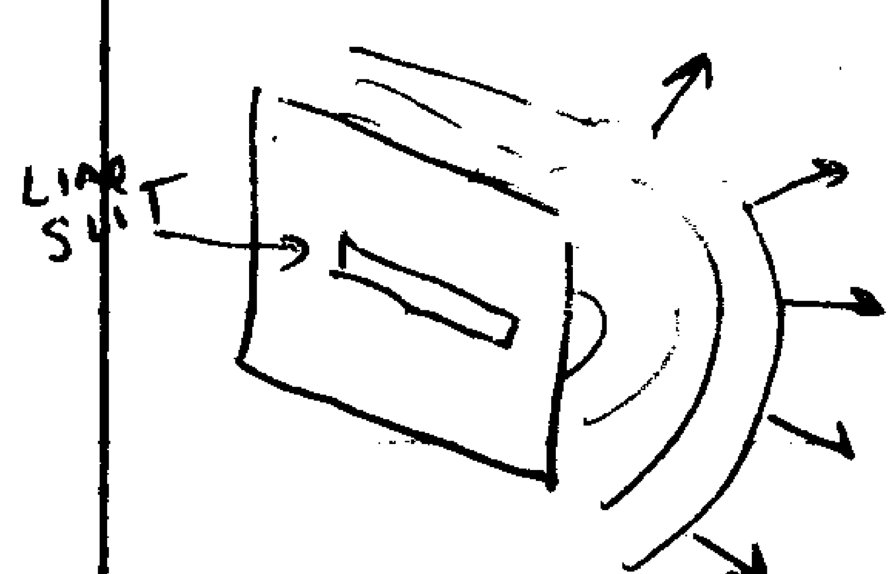


- Plane waves are sheets of uniform electrical field $\vec{E}$ propagating along $\vec{K}$

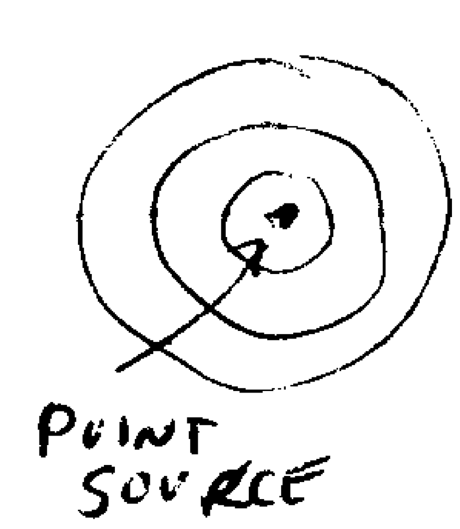
- WAVE SOLUTIONS NEED NOT BE PLANAR



Reflected / emitted from surface

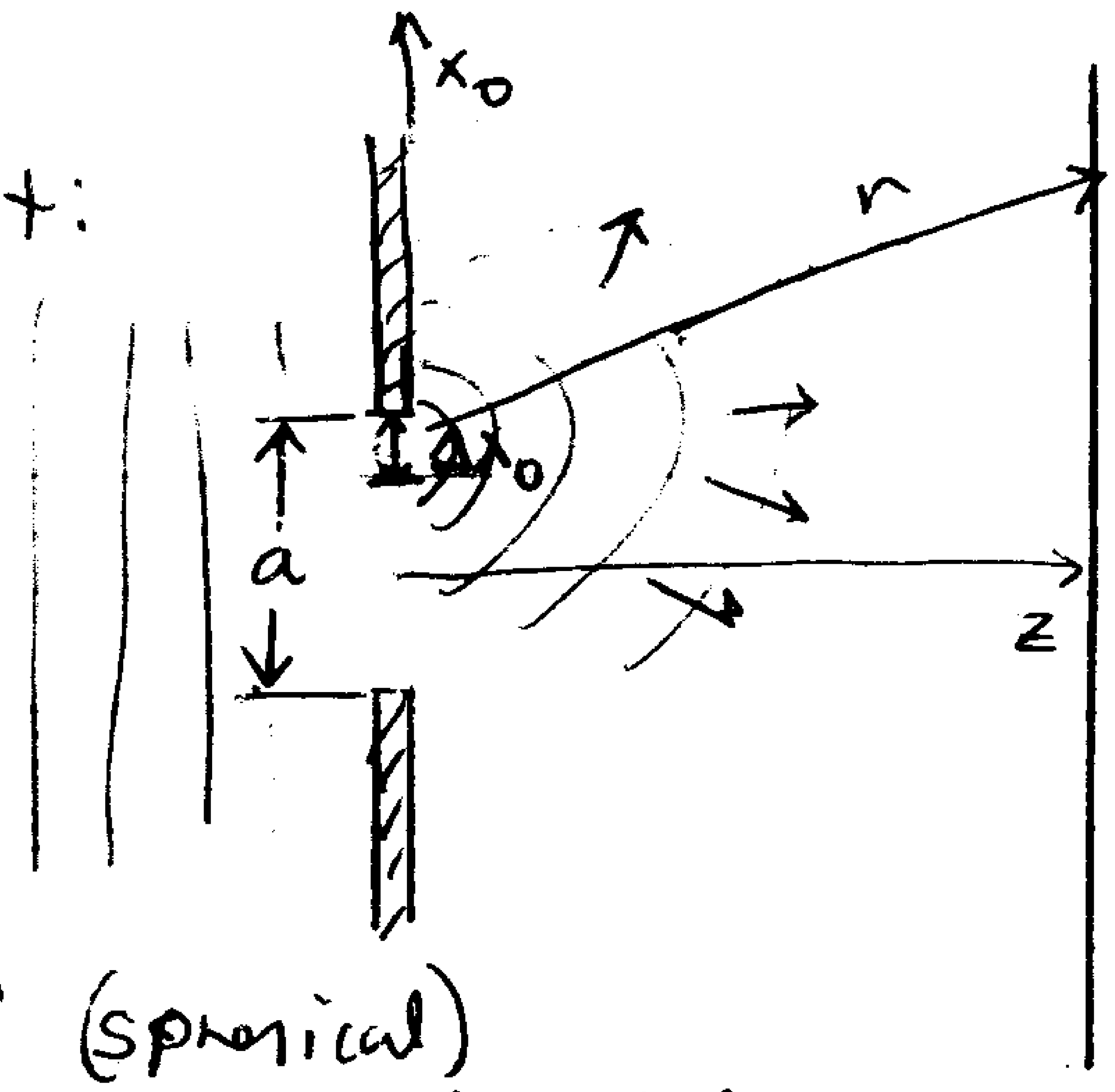


Cylindrical wave



Spherical waves.

Diffraction from a single Slit:



- Plane wave incident on slit of width a

- Use Huygen's Principle – wavefront emerging from slit is superposition of (spherical) waves Δu emitted from each differential section Δx_0 of slit.

$$u = \frac{U}{r} e^{ikr} \qquad \text{SPHERICAL WAVE}$$

– For point source at (x_0, y_0), $r = \sqrt{z^2 + (x-x_0)^2 + (y-y_0)^2}$

$$= z\sqrt{1 + \left(\frac{x-x_0}{z}\right)^2 + \left(\frac{y-y_0}{z}\right)^2}$$

– Fresnel APPROXIMATION

$$u \simeq \frac{U}{z} e^{ikr} \qquad \text{for } z > x_0, x, y_0, y$$

$$\Delta u(x,y,z) = \frac{U}{z} e^{ikz} \exp\left\{ i\frac{k}{2z}\left[(x-x_0)^2 + \cancel{(y-y_0)^2}\right]\right\}$$

Field amplitude from spherical wave.

$$u(x) = \frac{e^{ikz}}{z} \int_{-a/2}^{a/2} \exp\left[\frac{ik}{2z}(x-x_0)^2\right] dx_0$$

$$= \underbrace{\frac{e^{ikz}}{z} \exp\left(\frac{ik}{2z}x^2\right)}_{B} \int_{-a/2}^{a/2} \exp\left[\frac{ik}{2z}(-2xx_0 + \cancel{x_0^2})\right] dx_0$$

– Fraunhoffer approximation $\quad z \gg \frac{k x_0^2}{2}$

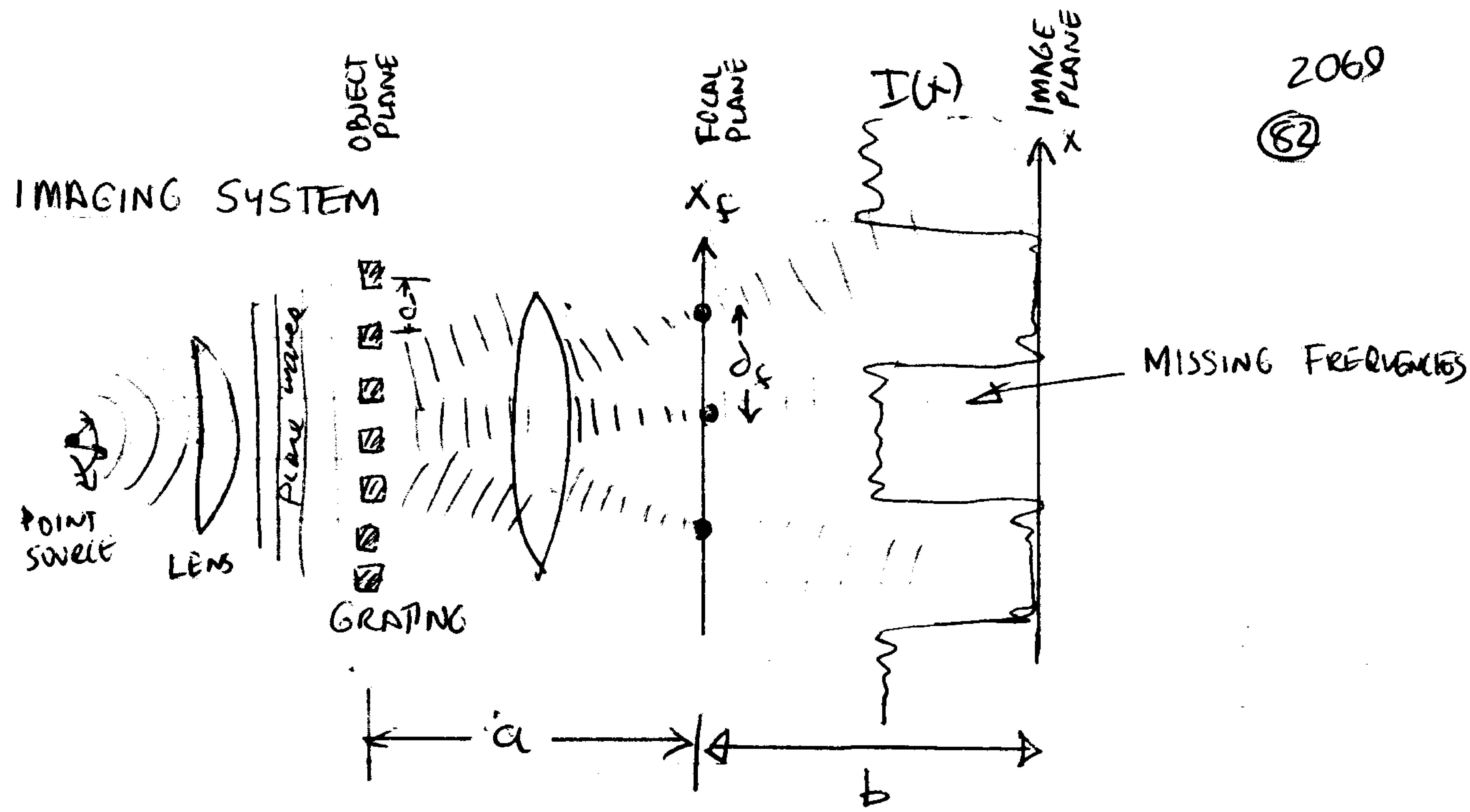
$$= \frac{B}{\cancel{z}}\left(\frac{\cancel{z}}{-ikx}\right) \exp\left(\frac{ikxx_0}{z}\right)\Bigg]_{-a/2}^{a/2} = \frac{-B}{ikx} \sin\left(\frac{kxa}{2z}\right) 2i$$

$$= \frac{2B}{kx} \sin\left(\frac{kxa}{2z}\right) = \frac{\lambda B}{\pi x} \sin\left(\frac{\pi a x}{z\lambda}\right)$$

– INTENSITY ON SCREEN

$$I(x) = |u(x)|^2 = |B|^2 \boxed{\frac{a^2 \sin^2\left(\frac{\pi a}{z\lambda}x\right)}{\left(\frac{\pi a x}{z\lambda}\right)^2}}$$

IMAGING SYSTEM

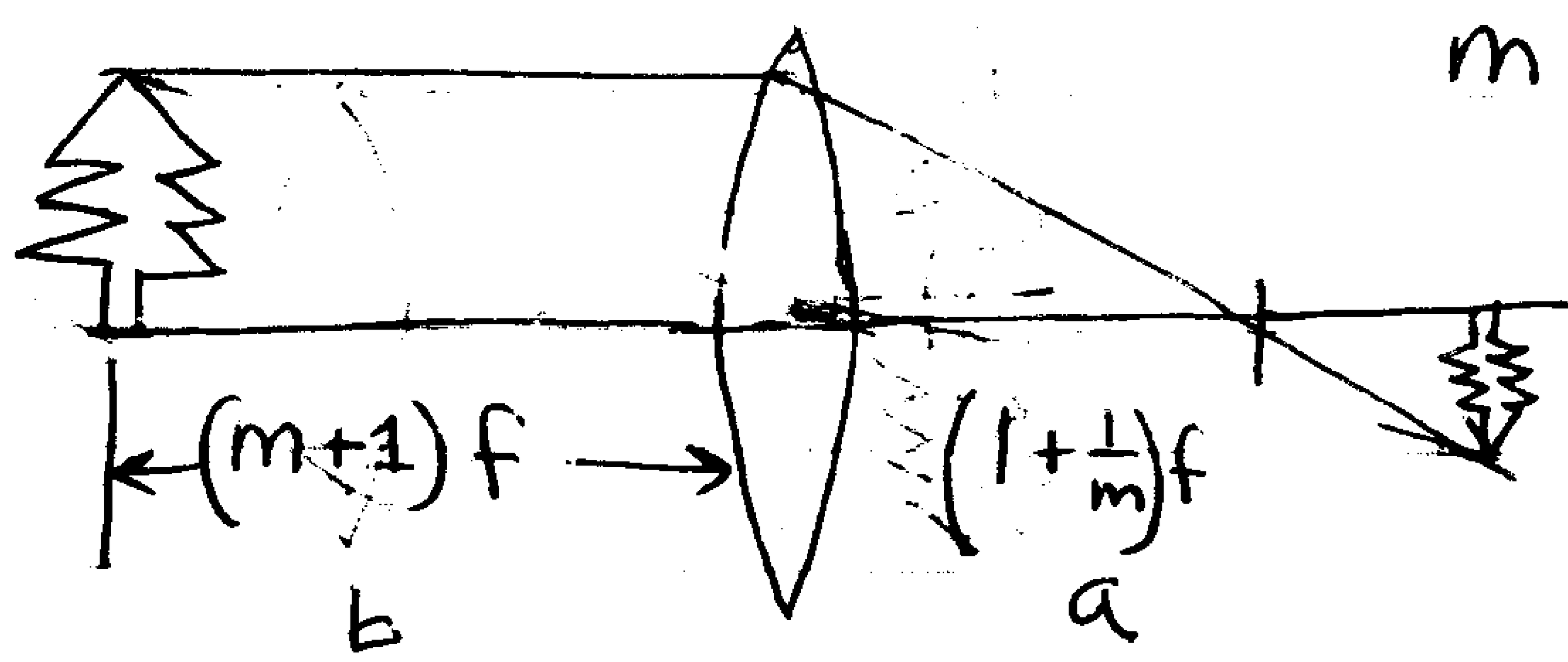


- Diffraction pattern (Fourier transform) of grating in focal plane $\quad f_0 = \frac{1}{d} \quad d_f = \lambda f f_0$

  as $d \downarrow$, $d_f \uparrow$

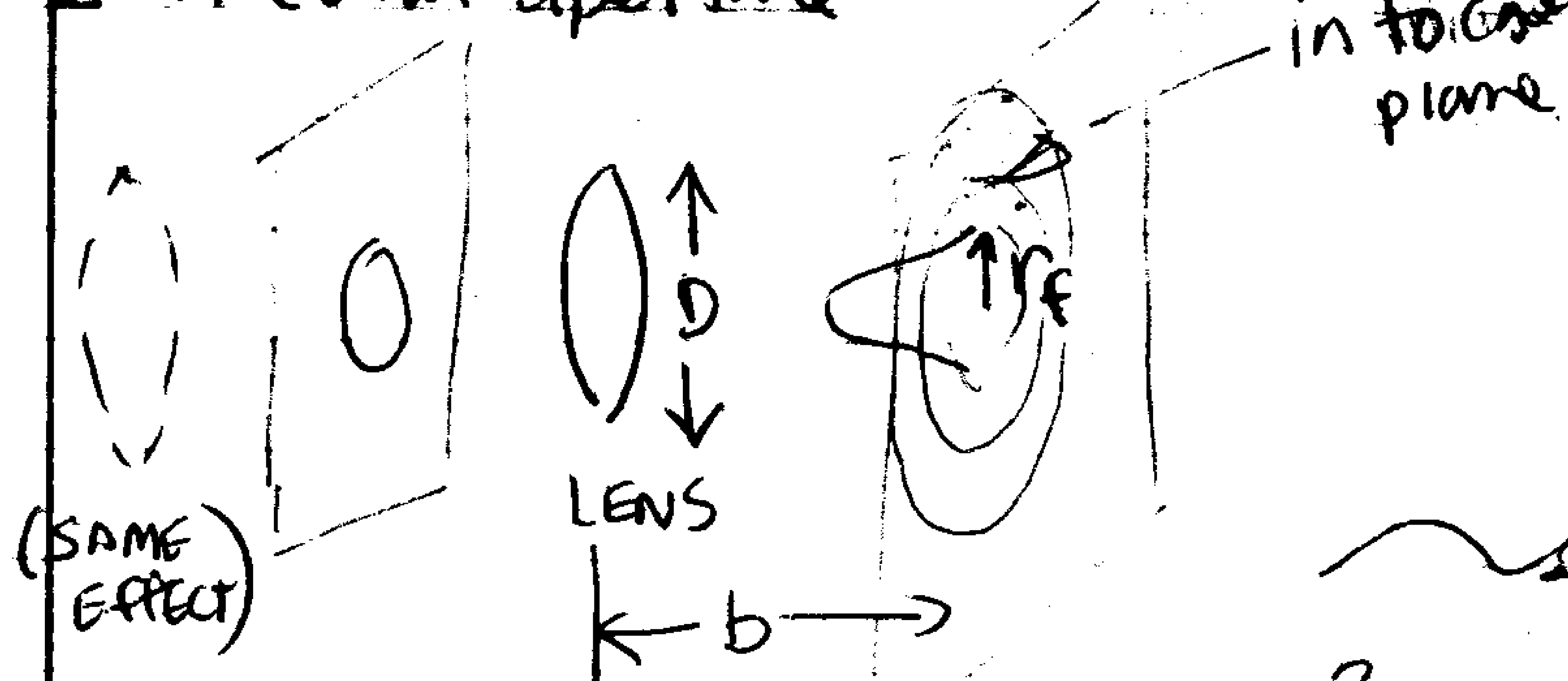
- Image of grating reassembled at image plane

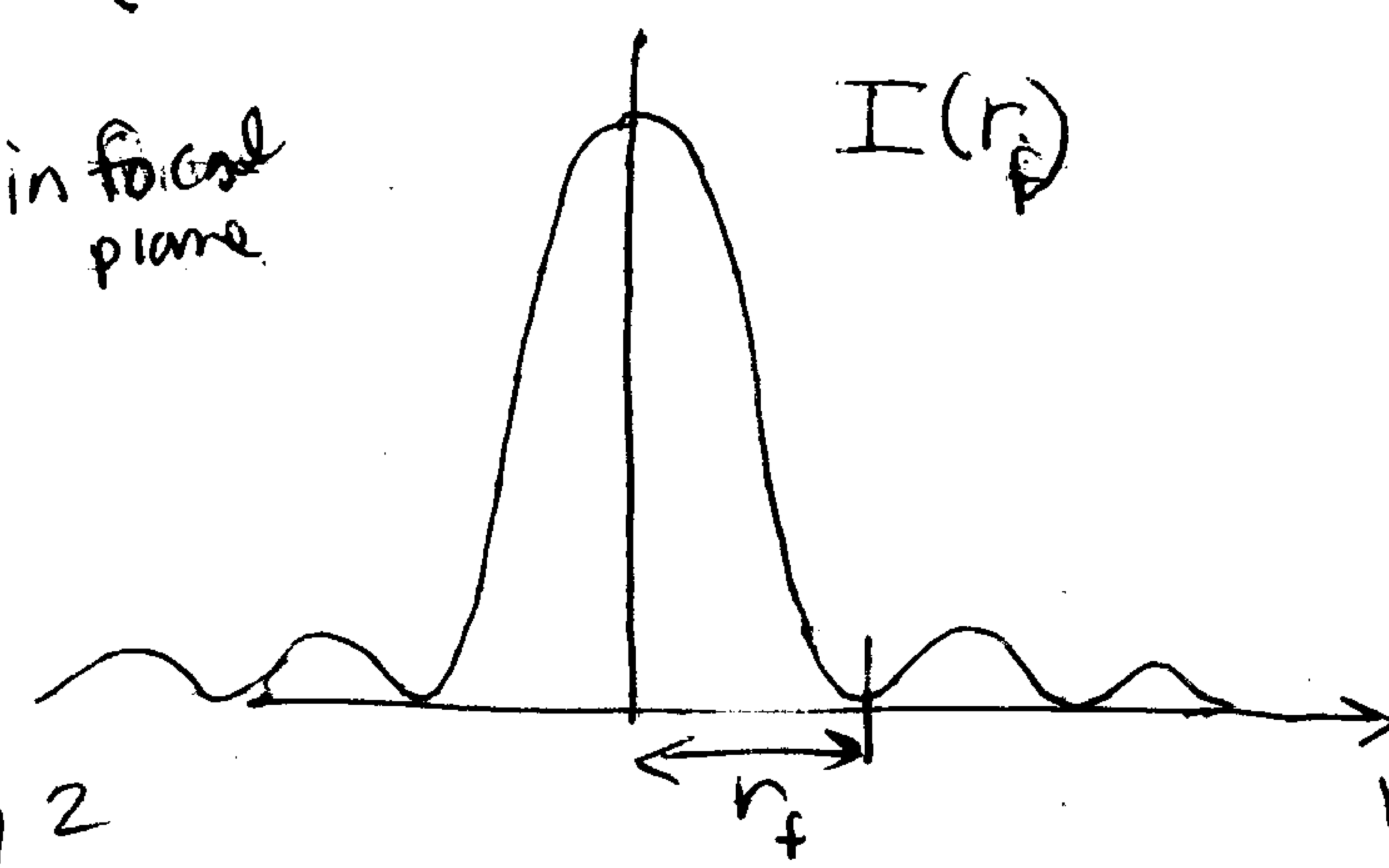
  m = magnification



$$\frac{b}{a} = \frac{m}{\cancel{m+1}}\cancel{(m+1)} = m$$

Circular aperture





$$I(r_f) = \left[\frac{J_1\left(\frac{kDr_f}{2b}\right)}{\frac{kDr_f}{2b}}\right]^2$$

Airy Pattern, J_1 = BESSEL FUNCTION

r_f = radius of airy disc

- Minimum in J_1 at $r = 3.833 = \frac{kDr_f}{2b} = \frac{2\pi D}{2b\lambda} r_f$

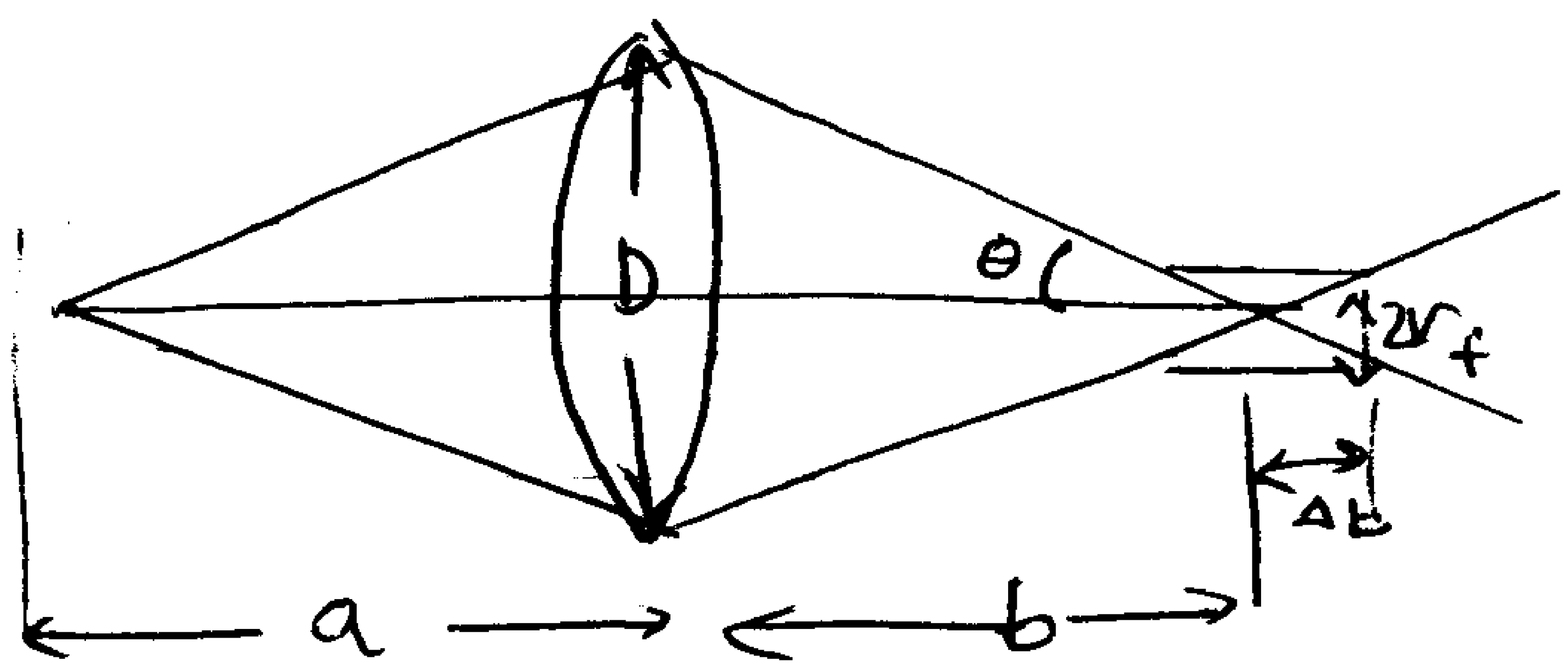
$$r_f = \frac{1.22\lambda b}{D} \quad \begin{matrix}\leftarrow \text{image distance} \\ \leftarrow \text{lens diameter}\end{matrix}$$

- Airy pattern is diffraction limited image of point source → indicator of resolution in optical system

— WANT TO HAVE $D\uparrow$, $\lambda\downarrow$, $b\downarrow$ TO INCREASE RESOLUTION

DEPTH OF FOCUS

- $b \pm \Delta b$ is region of acceptable focus $(r < r_f)$



(focus is always at best airy disc)

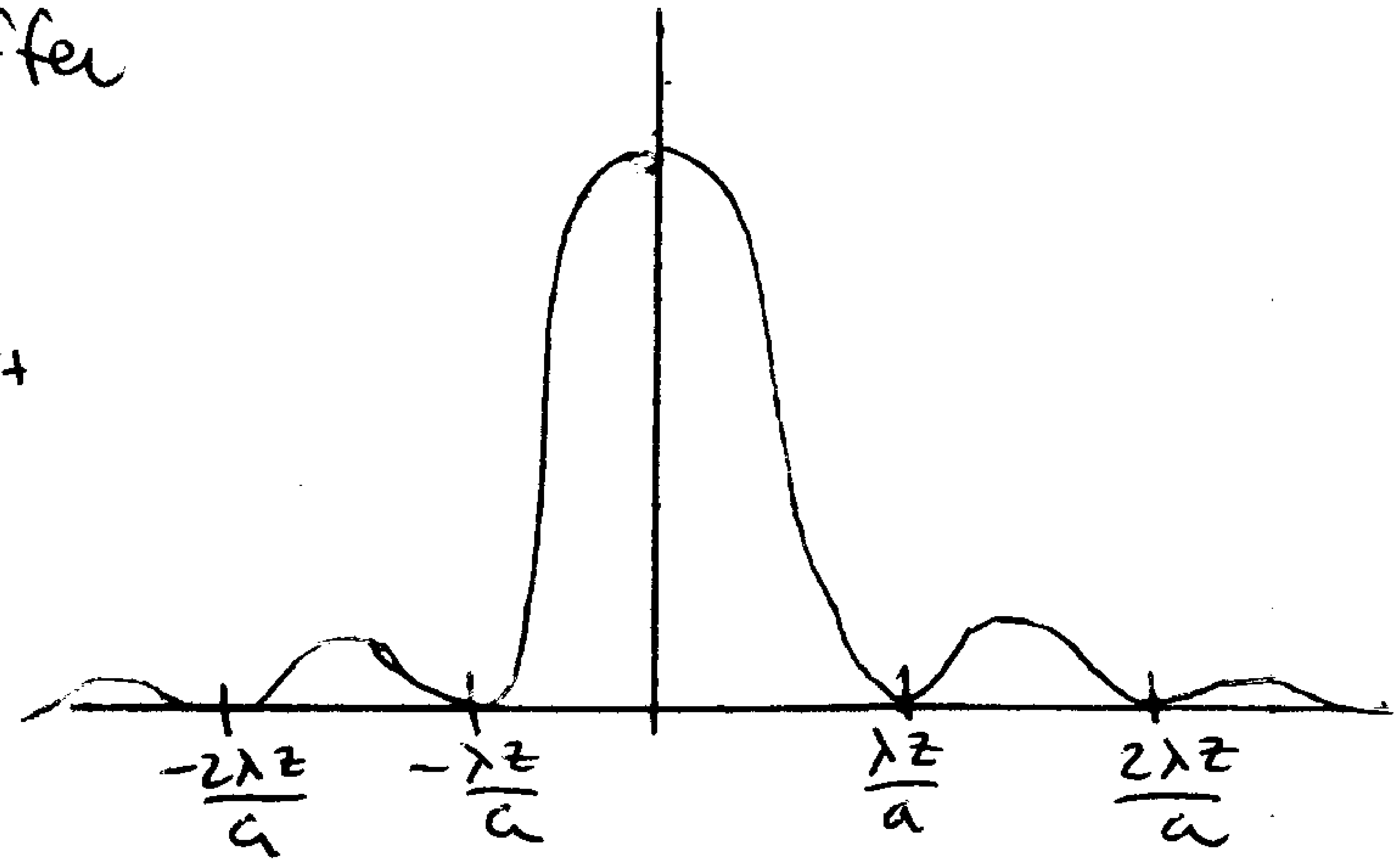
By SIMILAR TRIANGLES $\frac{D}{2b} = \frac{\Delta r}{\Delta b} = \frac{\Delta r_f}{\Delta b} = \frac{1.22\lambda b}{\Delta b D}$

$$\Delta b = 2(1.22)\lambda\left(\frac{b}{D}\right)^2 = 2.44\lambda\left(\frac{b}{D}\right)^2$$

$$NA = \frac{D}{f}, \quad \Delta b = 2.44\lambda\left(\frac{b}{f}\right)^2\frac{1}{(NA)^2} = 2.44\lambda(1+m^2)\frac{1}{NA^2}$$

- This is the Fraunhoffer diffraction pattern

$\Delta x \propto \frac{1}{a}$ ← SLIT WIDTH

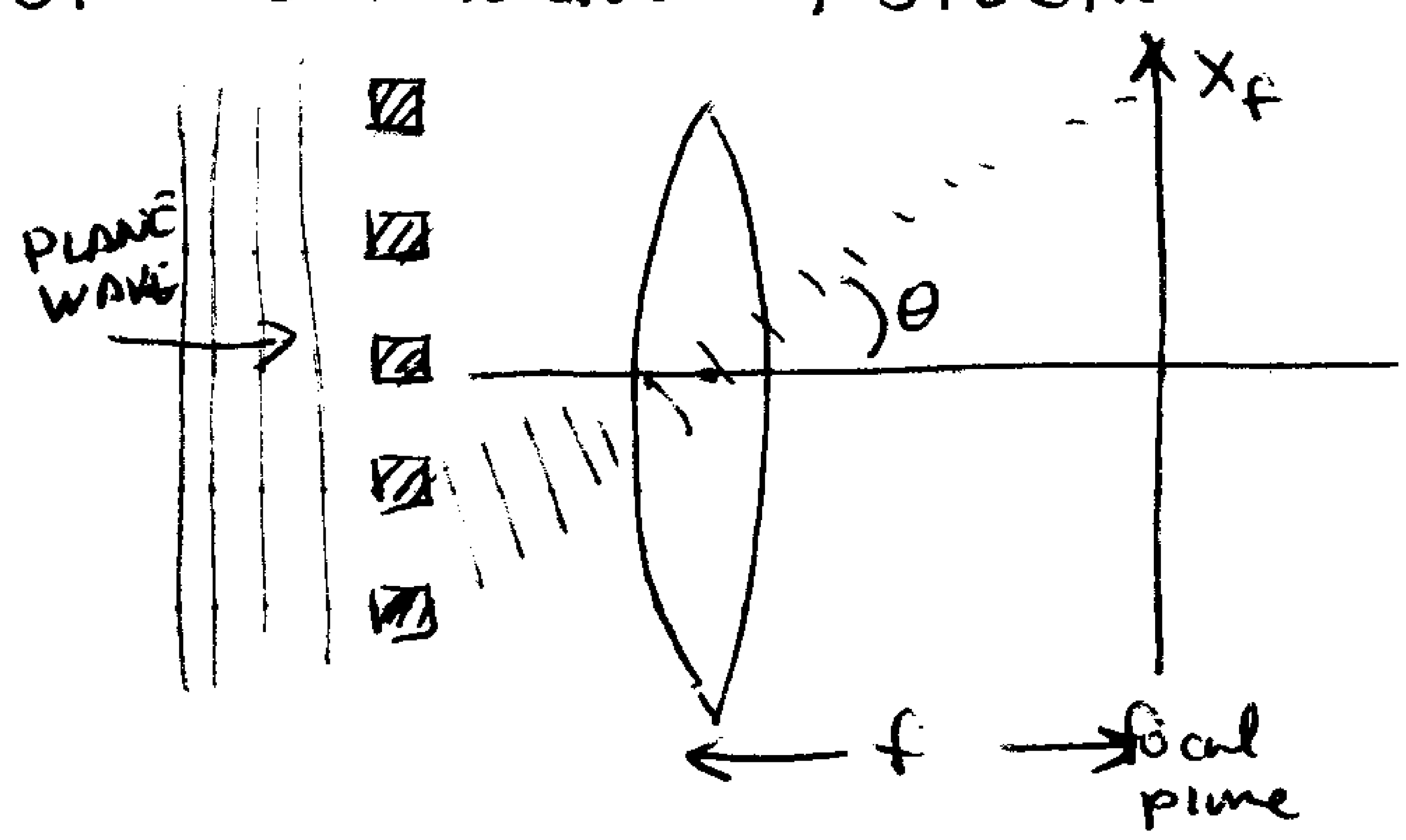


- More rigorously (Rayleigh-Sommerfeld diffraction formula)

$$u(P) = \frac{1}{i\lambda} \iint u(P_0) \frac{e^{ikr}}{r} \cos\Omega \, dS$$

angle
opening
incident field

SPATIAL FREQUENCY SPECTRUM

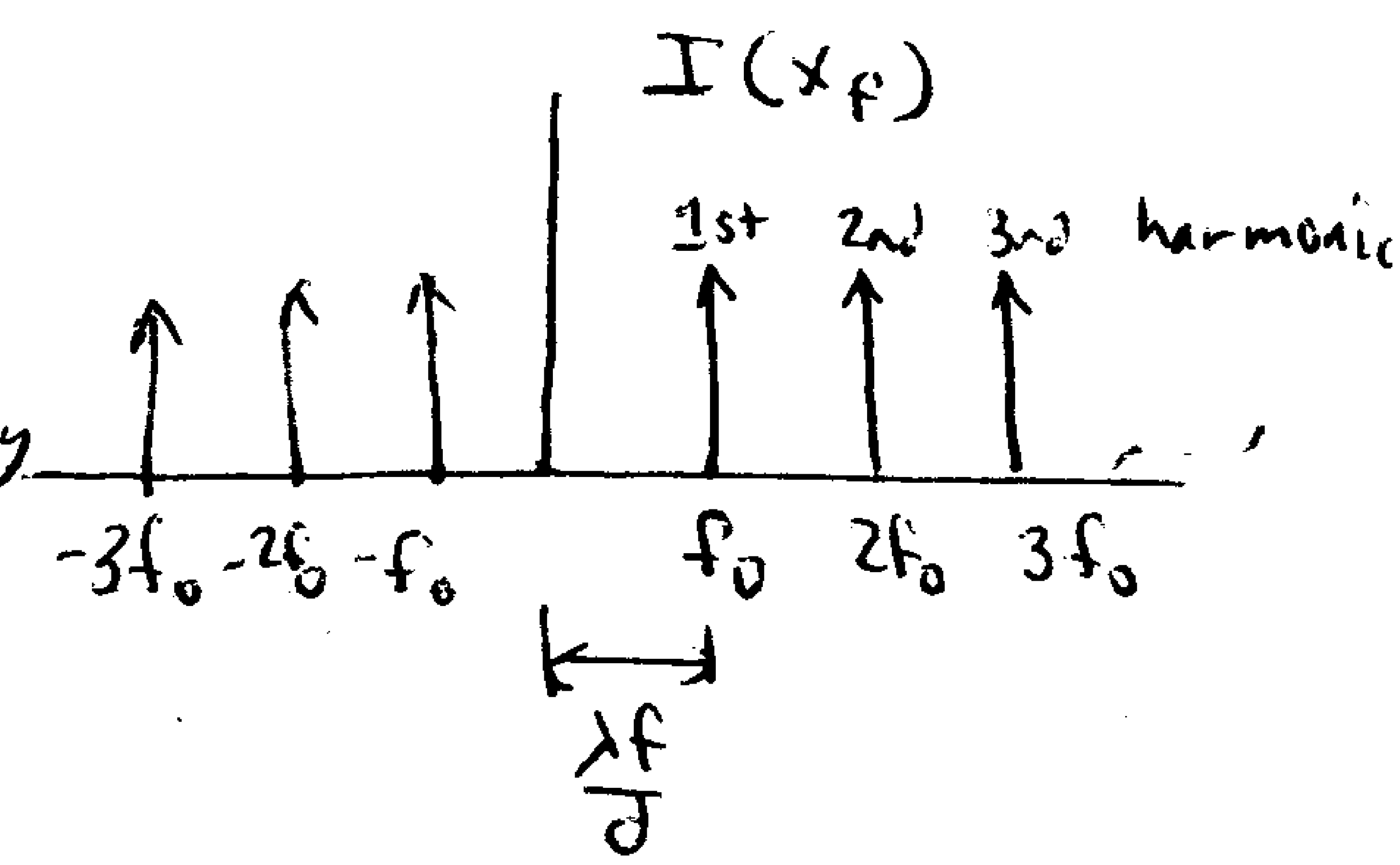


(Add Lens behind grating)

$x_f = f \tan\theta = \frac{f \sin\theta}{\cancel{\cos\theta}}$ → 1 small angle

Bragg condition $\sin\theta_n = \frac{n\lambda}{d}$

$x_f \simeq \frac{f n \lambda}{d} = f n \lambda f_0$ ← gratings f



- Focal plane contains frequency space representation of grating