BioE 2696/ECE 2695: Control Theory in Neuroscience
(3 Credits, Spring 2009)

Lecture 5: Control Theory II

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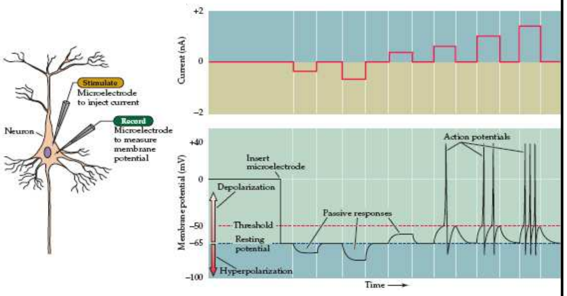
Outline

- Review of last lecture
- Time response of first-order systems
- Time response of second-order systems
- Time response specifications in design
- Frequency responses of systems

Review of last lecture

- Introduction to control theory
  - Open-loop and closed-loop
  - Feedback and feedforward (advantages and disadvantages)
  - Linear and nonlinear

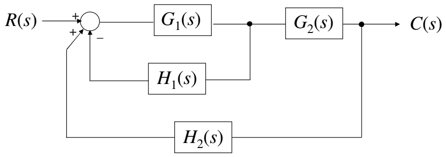


Copied from Dr. Aaron Batista's lecture notes

Review of last lecture

- Introduction to control theory
- Block diagrams

Exercise: Find out the transfer function of the following system:



Time response of first-order systems

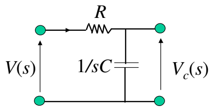
- First-order systems

dc gain

$$G(s)=\frac{C(s)}{R(s)}=\frac{b_0}{s+a_0}=\frac{K}{\tau s+1}$$

Time constant

  - Examples:

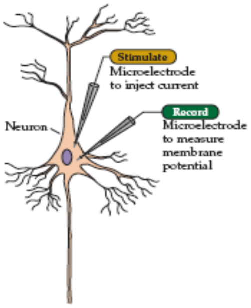


$$G(s)=\frac{V_C(s)}{V(s)}=\frac{1/(Cs)}{R+1/(Cs)}=\frac{1}{RCs+1}$$

Question: What does this circuit often used for?

Time response of first-order systems

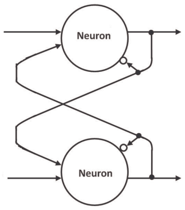
- First-order systems
  - Examples:



Neuron as leaky-integrator

Time response of first-order systems

- First-order systems
  - Examples:



Getting integrator from leaky-integrators

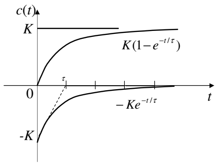
Time response of first-order systems

- First-order systems $G(s)=\frac{C(s)}{R(s)}=\frac{K}{\tau s+1}$
- Step response

$$R(s)=1/s,$$

$$C(s)=\frac{1}{s}\frac{K/\tau}{s+1/\tau}=\frac{K}{s}-\frac{K}{s+1/\tau},$$

$$c(t)=K(1-e^{-t/\tau}),\quad t>0$$



The limit of $c(t)$ as t goes to infinity is called the final value, or steady-state value of the response.

The parameter τ is called time constant; we may consider an exponential term to be zero after four time constants.

Time response of first-order systems

- First-order systems $G(s)=\frac{C(s)}{R(s)}=\frac{K}{\tau s+1}$
- Step response

$$R(s)=1/s,$$
$$C(s)=\frac{1}{s}\frac{K/\tau}{s+1/\tau}=\frac{K}{s}-\frac{K}{s+1/\tau},$$
$$c(t)=K-Ke^{-t/\tau},\quad t>0$$

Forced response or steady-state response

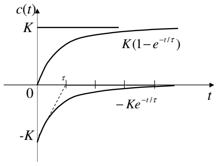
Natural response or transient response

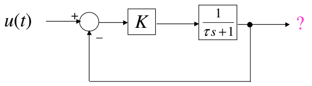
Time response of first-order systems

- First-order systems $G(s)=\frac{C(s)}{R(s)}=\frac{K}{\tau s+1}$
- Step response

$$c(t)=K(1-e^{-t/\tau}),\quad t>0$$

  - An example: realizing fast step response with a simple feedback controller





Time response of first-order systems

- First-order systems $G(s)=\frac{C(s)}{R(s)}=\frac{K}{\tau s+1}$
- Step response
- System dc gain
  - The system dc gain is the steady-state gain to a constant input for the case the output has a final value, and it is equal to the system transfer function evaluated at $s=0$ (why?)

Time response of first-order systems

- First-order systems
- Step response
- System dc gain

$$G(s)=\frac{C(s)}{R(s)}=\frac{K}{\tau s+1}$$

- Ramp response

$$R(s)=1/s^2,$$

$$C(s)=\frac{1}{s^2}\frac{K/\tau}{s+1/\tau}=\frac{K}{s^2}-\frac{K\tau}{s}+\frac{K\tau}{s+1/\tau},$$

$$c(t)=Kt-K\tau+K\tau\, e^{-t/\tau},\quad t>0$$

Steady-state response

$$c_{ss}(t)=Kt-K\tau$$

Time response of second-order systems

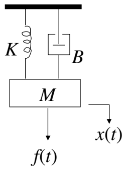
- Second-order systems

Natural frequency

$$G(s)=\frac{C(s)}{R(s)}=\frac{b_0}{s^2+a_1 s+a_0}=\frac{\omega_n^{\ 2}}{s^2+2\zeta\omega_n s+\omega_n^{\ 2}}$$

Damping ratio

  - Examples:



$$M\frac{d^2x}{dt^2}=f(t)-B\frac{dx}{dt}-Kx$$

$$G(s)=\frac{X(s)}{F(s)}=\frac{1}{Ms^2+Bs+K}$$

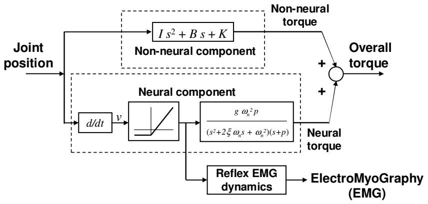
Time response of second-order systems

- Second-order systems
  - Examples:

An example from Dr. Ruiping Xia's research project (I am a collaborator)

Time response of second-order systems

- Second-order systems
  - Examples:



Time response of second-order systems

- Second-order systems $G(s) = \frac{\omega_n^{\,2}}{s^2 + 2\varsigma\omega_n s + \omega_n^{\,2}}$
- Step response
  - Case 1: ζ < 1 (underdamped), including ζ = 0 (undamped)

$$\boxed{c(t) = 1 - \frac{1}{\beta} e^{-\varsigma\omega_n t}\sin(\beta\omega_n t + \theta),}$$ where $\beta = \sqrt{1-\varsigma^2}$ and $\theta = \tan^{-1}(\beta/\varsigma)$

  - Case 2: ζ > 1 (overdamped)

$$\boxed{c(t) = 1 + k_1 e^{-t/\tau_1} + k_2 e^{-t/\tau_2},}$$ where $\tau_{1,2} = 1/\left(\varsigma\omega_n \pm \omega_n\sqrt{\varsigma^2 - 1}\right)$

  - Case 3: ζ = 1 (critically damped)

$$\boxed{c(t) = 1 + k_1 e^{-t/\tau} + k_2 t e^{-t/\tau},}$$ where $\tau = 1/\omega_n$

Time response of second-order systems

- Second-order systems $G(s) = \frac{\omega_n^{\,2}}{s^2 + 2\varsigma\omega_n s + \omega_n^{\,2}}$
- Step response

Case 1: ζ < 1 (underdamped)

$$\boxed{c(t) = 1 - \frac{1}{\beta} e^{-\varsigma\omega_n t}\sin(\beta\omega_n t + \theta),}$$

Case 2: ζ > 1 (overdamped)

$$\boxed{c(t) = 1 + k_1 e^{-t/\tau_1} + k_2 e^{-t/\tau_2},}$$

Case 3: ζ = 1 (critically damped)

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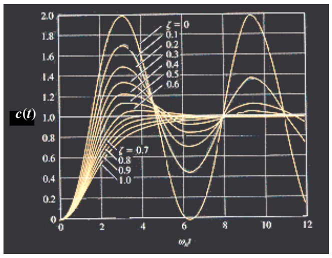
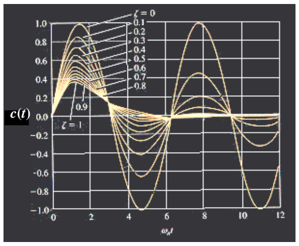


Examples of underdamped, overdamped, and critically damped responses:
(1) An animation (refer to the mov file)
(2) Heavy public doors with dashpots
(3) An example of sound 🔈

Time response of second-order systems

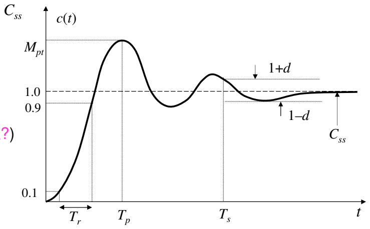
- Second-order systems $G(s)=\frac{\omega_n^{\ 2}}{s^2+2\varsigma\omega_n s+\omega_n^{\ 2}}$
- Step response
  - Case 1
  - Case 2
  - Case 3
  - Initial condition and impulse response

The initial condition excitation of higher-order systems cannot be modeled as simply as that of the first-order system; however, the impulse response of any system does give an indication of the nature of the initial-condition response, and thus the transient response



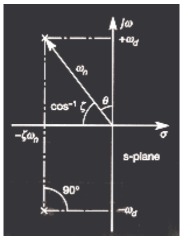
Time response of specifications in design

- Some parameters
  - Rise time, T_r
  - Peak value of the step response, M_{pt} ; time to reach it, T_p (how to calculate T_p?)
  - Steady state value, C_{ss}
  - Percent overshoot, $\frac{M_{pt}-C_{ss}}{C_{ss}}\times 100$
  - Settling time, T_s (how to calculate T_s?)



Time response of specifications in design

- Some parameters
- Time response and pole locations
  - The settling time is inversely related to the real part of the poles (the speed of response is increased by moving the poles to the left in the *s*-plane) $\boxed{T_s = k\tau = \frac{k}{\varsigma\omega_n}}$
  - Decreasing the angle $\cos^{-1}\zeta$ (increasing ζ) reduces the percent overshoot



Time response of specifications in design

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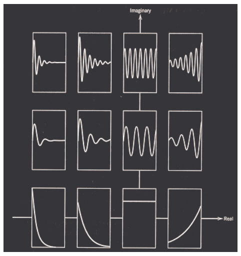


This picture shows how changing pole locations in the *s*-plane affects responses

Frequency response of systems

- Frequency response: steady-state response of systems to sinusoidal inputs

$$r(t) = A\cos\omega_1 t, \quad R(s) = \frac{As}{s^2+\omega_1^{\ 2}},$$

Assume that $\lim_{t\to\infty} c_g(t) = 0$

$$C(s) = G(s)R(s) = \frac{k_1}{s-j\omega_1} + \frac{k_2}{s+j\omega_1} + C_g(s)$$

$$k_1 = \frac{1}{2}AG(j\omega_1),\ k_2 = \frac{1}{2}AG(-j\omega_1), \quad \boxed{G(j\omega_1) = \left|G(j\omega_1)\right|e^{j\phi(\omega_1)}}$$

$$c_{ss}(t) = k_1 e^{j\omega_1 t} + k_2 e^{-j\omega_1 t} = A\left|G(j\omega_1)\right|\frac{e^{j(\omega_1 t+\phi(\omega_1))} + e^{-j(\omega_1 t+\phi(\omega_1))}}{2}$$

$$= A\left|G(j\omega_1)\right|\cos(\omega_1 t + \phi(\omega_1))$$

Frequency response of systems

- Frequency response: steady-state response of systems to sinusoidal inputs

$$r(t) = A\cos\omega_1 t, \quad G(j\omega_1) = |G(j\omega_1)|e^{j\phi(\omega_1)}$$

$$\boxed{c_{ss}(t) = A\,|G(j\omega_1)|\cos(\omega_1 t + \phi(\omega_1))}$$

  - The steady-state gain of a system for a sinusoidal input is the magnitude of the transfer function evaluation at $s = j\omega_1$, and the phase shift of the output sinusoid relative to the input sinusoid is the angle of $G(j\omega_1)$

Frequency response of systems

- Frequency response: steady-state response of systems to sinusoidal inputs
  - The steady-state gain of a system for a sinusoidal input is the *magnitude* of the transfer function evaluation at $s = j\omega_1$, and the *phase shift* of the output sinusoid relative to the input sinusoid is the angle of $G(j\omega_1)$
  - $G(j\omega)$ is defined as the frequency response function

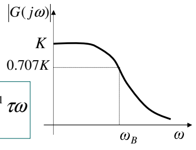
$$G(j\omega) = |G(j\omega)|e^{j\phi(\omega)}$$

Frequency response of systems

- Frequency response: steady-state response of systems to sinusoidal inputs
- Frequency response of first-order systems

$$G(s) = \frac{K}{\tau s + 1}$$

$$\boxed{|G(j\omega)| = \frac{K}{\left(1+\tau^2\omega^2\right)^{1/2}}, \quad \phi(\omega) = -\tan^{-1}\tau\omega}$$



  - System bandwidth, ω_B: The frequency at which the gain is equal to 1/sqrt(2) (approximately 0.707) times the gain at very low frequencies

Frequency response of systems

- Frequency response: steady-state response of systems to sinusoidal inputs
- Frequency response of first-order systems
- Frequency response of second-order systems

$$G(s)=\frac{\omega_n^{2}}{s^2+2\varsigma\omega_n s+\omega_n^{2}}=\frac{1}{(s/\omega_n)^2+2\varsigma(s/\omega_n)+1}$$

$$G(j\omega)=\frac{1}{[1-(\omega/\omega_n)^2]+j2\varsigma(\omega/\omega_n)}$$

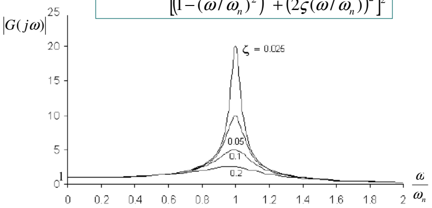
$$\boxed{|G(j\omega)|=\frac{1}{\left[\left(1-(\omega/\omega_n)^2\right)^2+\left(2\varsigma(\omega/\omega_n)\right)^2\right]^{\frac{1}{2}}}}$$

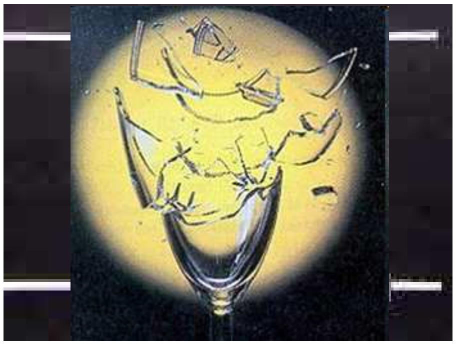
Question: What will happen if $\zeta = 0$ and $\omega = \omega_n$?

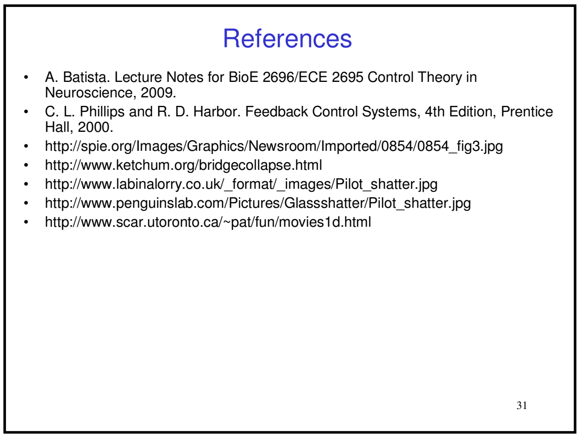
Frequency response of systems

- Frequency response: steady-state response of systems to sinusoidal inputs
- Frequency response of first-order systems
- Frequency response of second-order systems

$$\boxed{|G(j\omega)|=\frac{1}{\left[\left(1-(\omega/\omega_n)^2\right)^2+\left(2\varsigma(\omega/\omega_n)\right)^2\right]^{\frac{1}{2}}}}$$







References

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