BioE 2696/ECE 2695: Control Theory in Neuroscience
(3 Credits, Spring 2009)

Lecture 3: Control Theory I

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Outline

- Review of last lecture
- Introduction to control theory
- Transfer function (cont'd)
- Block diagrams
- Time response of first-order systems
- Time response of second-order systems

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Review of last lecture

- Meaning of systems
- Categories of systems
  - Memoryless system and system that has memory
  - Causal and noncausal systems
  - Linear and nonlinear systems
  - LTI system

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Review of last lecture

- Meaning of systems
- Categories of systems
- Laplace transform
- Representations of a system
  - Differential equations
  - State-space equations
  - Impulse response function and convolution
  - Transfer functions

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Introduction to control theory

- What is control system?
  - Generally speaking, a control system is a system that is used to realize a desired output or objective

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Introduction to control theory

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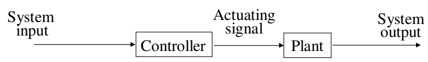
  - Generally speaking, a control system is a system that is used to realize a desired output or objective
  - Open-loop control systems



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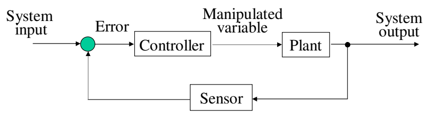
Introduction to control theory

- What is control system?
  - Generally speaking, a control system is a system that is used to realize a desired output or objective
  - Open-loop control systems
  - Closed-loop (or feedback) control systems



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Introduction to control theory

- What is control system?
  - Generally speaking, a control system is a system that is used to realize a desired output or objective
  - Open-loop control systems
  - Closed-loop (or feedback) control systems
  - Examples of feedback control systems
    - Clock and heart
    - Temperature control

Introduction to control theory

- What is control system?
  - Generally speaking, a control system is a system that is used to realize a desired output or objective
  - Open-loop control systems
  - Closed-loop (or feedback) control systems
  - Examples of feedback control systems
  - Advantages and disadvantages of feedback
    - Feedback allows high performance in the presence of uncertainty
    - Feedback allows the dynamics of a system to be modified
    - Feedback may create instability

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Introduction to control theory

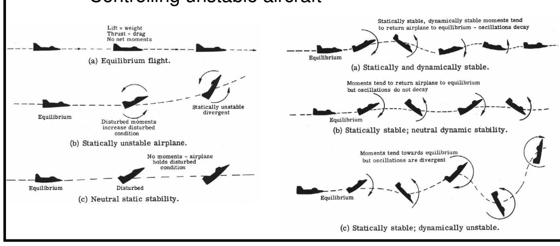
- What is control system?
- An example of control's application in aeronautics
  - Wright brothers realized that control was a key issue to enable flight

"Men already know how to construct wings or airplanes, which when driven through the air at sufficient speed, will not only sustain the weight of the wings themselves, but also that of the engine, and of the engineer as well. Men also know how to build engines and screws of sufficient lightness and power to drive these planes at sustaining speed … Inability to balance and steer still confronts students of the flying problem. … When this one feature has been worked out, the age of flying will have arrived, for all other difficulties are of minor importance."

— Quote from a lecture by Wilbur Wright to the Western Society of Engineers in 1901

Introduction to control theory

- What is control system?
- An example of control's application in aeronautics
  - Wright brothers realized that control was a key issue to enable flight
  - Controlling unstable aircraft



Introduction to control theory

- What is control system?
- An example of control's application in aeronautics
- Another example: human control of an inverted pendulum

A system with a right half plane pole p and a time delay T_d cannot be controlled unless the product pT_d is sufficient small. A simple rule of thumb is

$$pT_d < 0.16.$$

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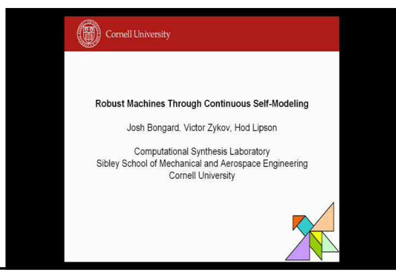
Introduction to control theory

- What is control system?
- An example of control's application in aeronautics
- Another example: human control of an inverted pendulum
- Biologically inspired control engineering



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Transfer functions (cont'd)

- Poles and zeros
  - Consider a transfer function which is a rational function in the complex variable s:

$$G(s) = \frac{N(s)}{D(s)} = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

where the numerator and denominator polynomials, $N(s)$ and $D(s)$, have real coefficients. The z_i's are the roots of the equation

$$N(s) = 0$$

and are defined to be the system zeros, and p_i's are the roots of the equation

$$D(s) = 0$$

and are defined to be the system poles

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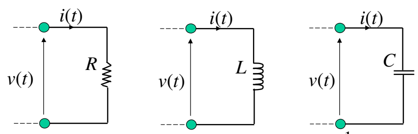
Transfer functions (cont'd)

- Poles and zeros
- Examples
  - Electrical circuits



$v(t) = i(t)R$ $\quad$ $v(t) = L\dfrac{di(t)}{dt}$ $\quad$ $v(t) = \dfrac{1}{C}\int_0^t i(\tau)d\tau$

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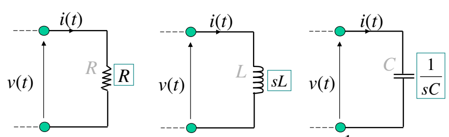
Transfer functions (cont'd)

- Poles and zeros
- Examples
  - Electrical circuits



$v(t) = i(t)R$ $\quad$ $v(t) = L\dfrac{di(t)}{dt}$ $\quad$ $v(t) = \dfrac{1}{C}\int_0^t i(\tau)d\tau$

$\dfrac{V(s)}{I(s)} = R$ $\quad$ $\dfrac{V(s)}{I(s)} = sL$ $\quad$ $\dfrac{V(s)}{I(s)} = \dfrac{1}{sC}$

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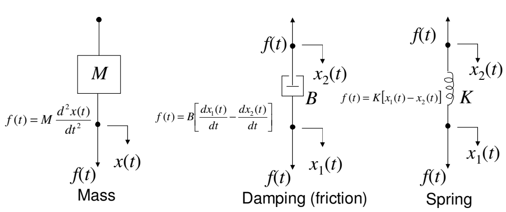
Transfer functions (cont'd)

- Poles and zeros
- Examples
  - Electrical circuits
  - Mechanical translational systems



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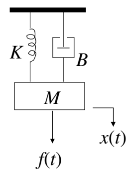
Transfer functions (cont'd)

- Poles and zeros
- Examples
  - Electrical circuits
  - Mechanical translational systems



$$M\frac{d^2x}{dt^2} = f(t) - B\frac{dx}{dt} - Kx$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

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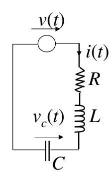
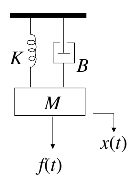
Transfer functions (cont'd)

- Poles and zeros
- Examples
  - Electrical circuits
  - Mechanical translational systems
  - Analogous systems



$$G(s) = \frac{V_C(s)}{V(s)} = \frac{1}{LCs^2 + RCs + 1}$$



$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

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Transfer functions (cont'd)

- Poles and zeros
- Examples
  - Electrical circuits
  - Mechanical translational systems
  - Analogous systems
  - Commonly seen transfer functions

k | s | $\frac{1}{s}$ | $\frac{1}{s+p}$ | $\frac{K}{\tau s+1}$ | $\frac{s+z}{s+p}$

$\frac{\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$ | $K_p + \frac{K_I}{s} + \frac{K_D s}{\tau s + 1}$ | e^{-Ts}

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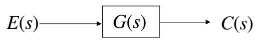
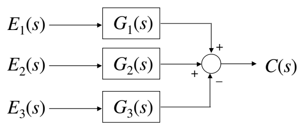
Block diagrams

- The transfer function relationship $C(s) = G(s)\,E(s)$ can be graphically denoted through a block diagram



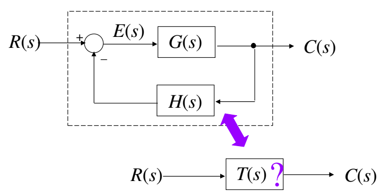
- Summing junction in a block diagram



$$C(s) = G_1(s)\,E_1(s) + G_2(s)\,E_2(s) - G_3(s)\,E_3(s)$$

Block diagrams

- The transfer function relationship can be graphically denoted through a block diagram
- Summing junction in a block diagram
- Example:



Block diagrams

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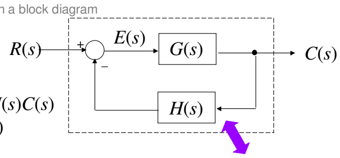
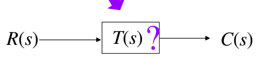
- Example:



$$\begin{cases} E(s) = R(s) - H(s)C(s) \\ C(s) = G(s)E(s) \end{cases}$$

$$C(s) = \frac{G(s)}{1 + G(s)H(s)} R(s)$$



$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Block diagrams

- The transfer function relationship can be graphically denoted through a block diagram
- Summing junction in a block diagram
- Example
- Finding system transfer functions involves solving simultaneously algebra equations
  - by eliminating variables
  - by Cramer's rules
  - by inverse matrix procedures

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Time response of first-order systems

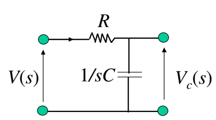
- First-order systems

$$G(s)=\frac{C(s)}{R(s)}=\frac{b_0}{s+a_0}=\frac{K}{\tau s+1}$$

dc gain

Time constant

  - An example:



$$G(s)=\frac{V_C(s)}{V(s)}=\frac{1/(Cs)}{R+1/(Cs)}=\frac{1}{RCs+1}$$

Question: What does this circuit often used for?

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Time response of first-order systems

- First-order systems

$$G(s)=\frac{C(s)}{R(s)}=\frac{b_0}{s+a_0}=\frac{K}{\tau s+1}$$

  - An example
  - Initial conditions

$$\left(s+\frac{1}{\tau}\right)C(s)=\frac{K}{\tau}R(s) \quad\rightarrow\quad \frac{dc(t)}{dt}+\frac{1}{\tau}c(t)=\frac{K}{\tau}r(t)$$

With zero initial condition

$$sC(s)-c(0)+\frac{1}{\tau}C(s)=\frac{K}{\tau}R(s)$$

$$C(s)=\frac{c(0)}{s+(1/\tau)}+\frac{K}{\tau s+1}R(s)$$

$$=\frac{K}{\tau s+1}\left(R(s)+c(0)\frac{\tau}{K}\right)$$

With initial term

Initial condition is equivalent to an input of impulse function

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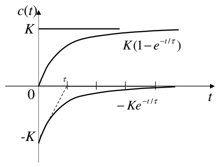
Time response of first-order systems

- First-order systems $G(s)=\frac{C(s)}{R(s)}=\frac{K}{\tau s+1}$
- Step response

$$R(s)=1/s,$$
$$C(s)=\frac{1}{s}\frac{K/\tau}{s+1/\tau}=\frac{K}{s}-\frac{K}{s+1/\tau},$$
$$c(t)=K(1-e^{-t/\tau}),\quad t>0$$



The limit of $c(t)$ as t goes to infinity is called the final value, or steady-state value of the response.

The parameter τ is called time constant; we may consider an exponential term to be zero after four time constants.

Time response of first-order systems

- First-order systems $G(s)=\frac{C(s)}{R(s)}=\frac{K}{\tau s+1}$
- Step response

$$R(s)=1/s,$$
$$C(s)=\frac{1}{s}\frac{K/\tau}{s+1/\tau}=\frac{K}{s}-\frac{K}{s+1/\tau},$$
$$c(t)=K-Ke^{-t/\tau},\quad t>0$$

Forced response or steady-state response

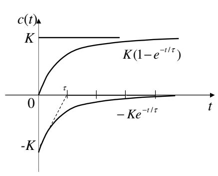
Natural response or transient response

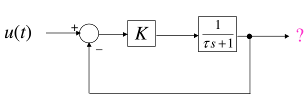
Time response of first-order systems

- First-order systems $G(s)=\frac{C(s)}{R(s)}=\frac{K}{\tau s+1}$
- Step response

$$c(t)=K(1-e^{-t/\tau}),\quad t>0$$

– An example: realizing fast step response with a simple feedback controller





Time response of first-order systems

- First-order systems $G(s) = \frac{C(s)}{R(s)} = \frac{K}{\tau s + 1}$
- Step response
- System dc gain
  - The system dc gain is the steady-state gain to a constant input for the case the output has a final value, and it is equal to the system transfer function evaluated at $s = 0$ (why?)

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Time response of first-order systems

- First-order systems $G(s) = \frac{C(s)}{R(s)} = \frac{K}{\tau s + 1}$
- Step response
- System dc gain
- Ramp response

$$R(s) = 1/s^2,$$

$$C(s) = \frac{1}{s^2}\frac{K/\tau}{s+1/\tau} = \frac{K}{s^2} - \frac{K\tau}{s} + \frac{K\tau}{s+1/\tau},$$

$$c(t) = Kt - K\tau + K\tau\, e^{-t/\tau}, \quad t > 0$$

Steady-state response

$$c_{ss}(t) = Kt - K\tau$$

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Time response of second-order systems

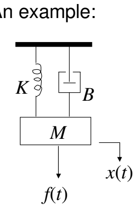
- Second-order systems

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_0}{s^2 + a_1 s + a_0} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Natural frequency (ω_n)

Damping ratio (ζ)

  - An example:



$$M\frac{d^2x}{dt^2} = f(t) - B\frac{dx}{dt} - Kx$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

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Time response of second-order systems

- Second-order systems $G(s)=\dfrac{\omega_n^2}{s^2+2\varsigma\omega_n s+\omega_n^2}$
- Step response
  - Case 1: $\zeta<1$ (underdamped), including $\zeta=0$ (undamped)

$$c(t)=1-\frac{1}{\beta}e^{-\varsigma\omega_n t}\sin(\beta\omega_n t+\theta),\quad \text{where } \beta=\sqrt{1-\varsigma^2} \text{ and } \theta=\tan^{-1}(\beta/\varsigma)$$

  - Case 2: $\zeta>1$ (overdamped)

$$c(t)=1+k_1e^{-t/\tau_1}+k_2e^{-t/\tau_2},\quad \text{where } \tau_{1,2}=1/\left(\varsigma\omega_n\pm\omega_n\sqrt{\varsigma^2-1}\right)$$

  - Case 3: $\zeta=1$ (critically damped)

$$c(t)=1+k_1e^{-t/\tau}+k_2te^{-t/\tau},\quad \text{where } \tau=1/\omega_n$$

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Time response of second-order systems

- Second-order systems $G(s)=\dfrac{\omega_n^2}{s^2+2\varsigma\omega_n s+\omega_n^2}$
- Step response

Case 1: $\zeta<1$ (underdamped)

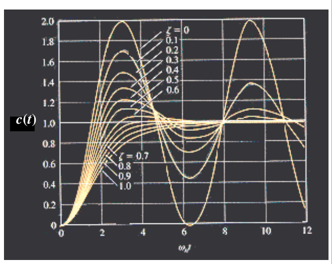
$$c(t)=1-\frac{1}{\beta}e^{-\varsigma\omega_n t}\sin(\beta\omega_n t+\theta),$$

Case 2: $\zeta>1$ (overdamped)

$$c(t)=1+k_1e^{-t/\tau_1}+k_2e^{-t/\tau_2},$$

Case 3: $\zeta=1$ (critically damped)

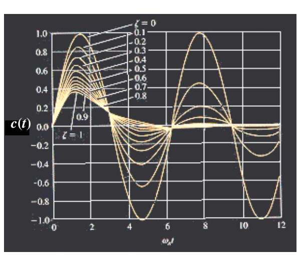
$$c(t)=1+k_1e^{-t/\tau}+k_2te^{-t/\tau},$$



Time response of second-order systems

- Second-order systems $G(s)=\dfrac{\omega_n^2}{s^2+2\varsigma\omega_n s+\omega_n^2}$
- Step response
  - Case 1
  - Case 2
  - Case 3
  - Initial condition and impulse response

The initial condition excitation of higher-order systems cannot be modeled as simply as that of the first-order system; however, the impulse response of any system does give an indication of the nature of the initial-condition response, and thus the transient response



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