BioE 2696/ECE 2695: Control Theory in Neuroscience
(3 Credits, Spring 2009)

Lecture 2: Signals and Systems II

January 7, 2009

Zhi-Hong Mao
Assistant Professor of ECE and Bioengineering
University of Pittsburgh, Pittsburgh, PA

1

Outline

- Review of last lecture
- Useful online materials for control theory
- Meaning of systems
- Categories of systems
- Representations of a system
- Laplace transform
- Transfer functions

2

Review of last lecture

- Signal, noise, and information
  - Information transmission rate of continuous movements and discrete movements
- Fourier transform

Forward transform $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

Inverse transform $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{+j\omega t}d\omega$

Question: Are there any requirements on $x(t)$?

3

Useful online materials for control theory

- K. J. Astrom and R. M. Murray. Feedback Systems: An Introduction for Scientists and Engineers. Manuscript, 2007. Available online: http://www.cds.caltech.edu/~murray/books/AM05/wiki/index.php?title=Main_Page
- D. S. Bernstein. A Student's Guide to Classical Control. Available online: http://www.engr.pitt.edu/electrical/faculty-staff/mao/2646/Lectures/classic_control.pdf
- M. Dahleh, M. A. Dahleh, and G. Verghese. Lecture Notes for 6.241 Dynamic Systems and Control. MIT, 2003. Available online: http://ocw.mit.edu/OcwWeb/Electrical-Engineering-and-Computer-Science/6-241Fall2003/LectureNotes/index.htm
- J. Deyst and K. Willcox. Lecture Notes for 16.060 Principles of Automatic Control. MIT, 2003. Available online: http://ocw.mit.edu/OcwWeb/Aeronautics-and-Astronautics/16-06Fall2003/LectureNotes/index.htm
- ECE 1673: http://www.engr.pitt.edu/electrical/faculty-staff/mao/1673/
- ECE 2646: http://www.engr.pitt.edu/electrical/faculty-staff/mao/2646/
- ECE 2695: http://www.engr.pitt.edu/electrical/faculty-staff/mao/2695/

4

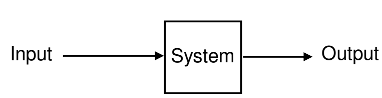
Meaning of systems

- Definition
  - A system processes signals
  - Examples in neural systems:
    - Passive neural membrane: transforming injected current into membrane potential
    - Cochlea: transforming sound into cochlear microphonic (i.e., electrical potential generated in the hair cells of the organ of Corti in response to acoustic stimulation)
    - Optics of the eye: transforming visual stimulus into retinal image
    - Retinal ganglion cell: transforming contrast into firing rate
    - Human

5

Meaning of systems

- Definition
  - A system processes signals
  - Examples in neural systems
  - Mathematically speaking, a system is a function mapping (or transforming) input signals into output signals



6

Categories of systems

- Memoryless system and system that has memory
  - A system is called a memoryless system if its output $y(t_0)$ depends only on the input applied at t_0; it is independent of the input applied before or after t_0

Question: Are the following systems memoryless or having memory?

$$y(t) = u(t)^2,$$
$$y(t) = u(t-1),$$
$$y(t) = \int_0^t u(t)dt,$$
$$y(t) = du/dt.$$

7

Categories of systems

- Memoryless system and system that has memory
  - A system is called a memoryless system if its output $y(t_0)$ depends only on the input applied at t_0; it is independent of the input applied before or after t_0

Questions:
Is the cochlear a memoryless system?
Is the visual system a memoryless system?
Is a person in typing a memoryless system?

8

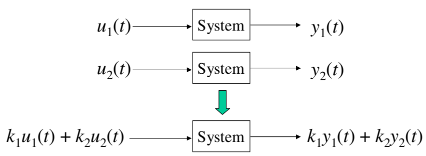
Categories of systems

- Memoryless system and system that has memory
- Causal and noncausal system
  - A system is called a causal or nonanticipatory system if its current output depends on past and current but not future input
  - If a system is noncausal, then its current output will depend on future input (no physical system has such capability!)

9

Categories of systems

- Memoryless system and system that has memory
- Causal and noncausal system
- Linear and nonlinear systems
  - A system is linear if superposition applies

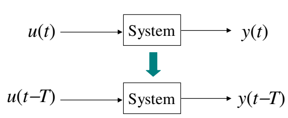


Question: Is $y(t) = u(t) + 2$ a linear system?

10

Categories of systems

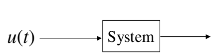
- Memoryless system and system that has memory
- Causal and noncausal system
- Linear and nonlinear systems
  - A system is linear if superposition applies
  - A system is linear time-invariant (LTI) if it is linear and satisfies that whether we apply an input to the system now or T seconds from now, the output will be identical, except for a time delay of the T seconds



11

Representations of a system

- Differential equations
  - Most dynamical systems in control engineering can be modeled as differential equations



$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_m\frac{d^m u(t)}{dt^m} + \cdots + b_1\frac{du(t)}{dt} + b_0 u(t)$$

12

Representations of a system

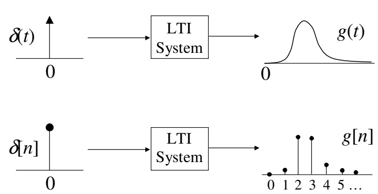
- Differential equations
- State-space equations
  - A system is modeled as a set of first-order differential equations (representation of the dynamics of an n th-order system using n first-order differential equations)

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

13

Representations of a system

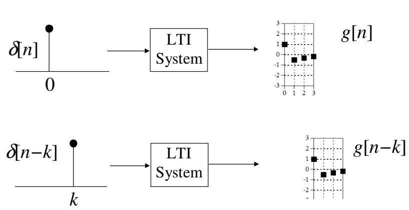
- Differential equations
- State-space equations
- Impulse response functions and convolution



14

Representations of a system

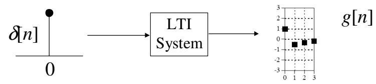
- Differential equations
- State-space equations
- Impulse response functions and convolution



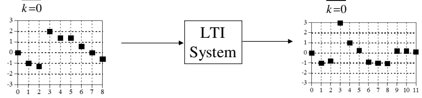
15

Representations of a system

- Differential equations
- State-space equations
- Impulse response functions and convolution



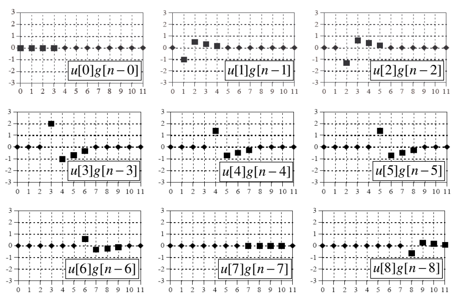
$$u[n] = \sum_{k=0}^{n} u[k]\delta[n-k] \qquad y[n] = \sum_{k=0}^{n} u[k]g[n-k]$$



16

Representations of a system

- Differential equations
- State-space equations
- Impulse response functions and convolution



17

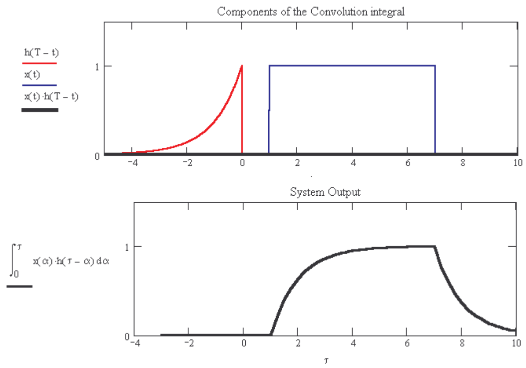
Representations of a system

- Differential equations
- State-space equations
- Impulse response functions and convolution

$$y[n] = \sum_{k=0}^{n} u[k]g[n-k] \longleftrightarrow y[n] = u[n] * g[n]$$

$$y(t) = \int_{0}^{t} u(\tau)g(t-\tau)d\tau \longleftrightarrow y(t) = u(t) * g(t)$$

18



19

Representations of a system

- Differential equations
- State-space equations
- Impulse response functions and convolution

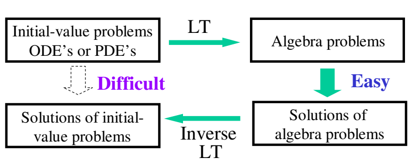
"For the modern EE, the lure of the Laplace transform is its ability to map the complicated operation of convolution into multiplication. This integral has for decades driven electrical engineering undergraduates to contemplate theology either for salvation or as an alternative career."

— Paul J. Nahin

20

Representations of a system

- Differential equations
- State-space equations
- Impulse response functions and convolution



21

Laplace transform

- The Laplace transform of a function $f(t)$ is defined as

$$F(s) = L[f(t)] = \int_0^{\infty} f(t)e^{-st}dt$$

22

Laplace transform

- The Laplace transform of a function $f(t)$ is defined as

$$F(s) = L[f(t)] = \int_0^{\infty} f(t)e^{-st}dt$$ Laplace transform

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$ Fourier transform

  – A little history: Euler, Lagrange, and Laplace

Question: Laplace transform and Fourier transform, which one was developed earlier?

23

Laplace transform

- The Laplace transform of a function $f(t)$ is defined as

$$F(s) = L[f(t)] = \int_0^{\infty} f(t)e^{-st}dt$$ Laplace transform

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$ Fourier transform

  – A little history: Euler, Lagrange, and Laplace
  – Comparison with Fourier transform
    - The step function has a Laplace transform, but not a Fourier transform
    - While the Fourier transform is useful in finding the steady-state output of a linear system in response to a periodic input, the Laplace transform can provide both the steady-state and transient responses for periodic and aperiodic inputs

24

Laplace transform

- The Laplace transform of a function $f(t)$ is defined as

$$F(s) = L[f(t)] = \int_0^{\infty} f(t)e^{-st}dt$$

- The inverse Laplace transform is given by

$$f(t) = L^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$$

25

Laplace transform

- The Laplace transform of a function $f(t)$ is defined as

$$F(s) = L[f(t)] = \int_0^{\infty} f(t)e^{-st}dt$$

- The inverse Laplace transform is given by

$$f(t) = L^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-\infty}^{\sigma+\infty} F(s)e^{st}ds$$

  – We seldom use the above equation to calculate an inverse Laplace transform; instead we use the equation of Laplace transform to construct a table of transforms for useful time functions. Then we use the table to find the inverse transform

26

Laplace transform table

TIME DOMAIN		FREQUENCY DOMAIN
$\delta(t)$	unit impulse	1
A	step	$\frac{A}{s}$
t	ramp	$\frac{1}{s^2}$
t^2		$\frac{2}{s^3}$
$t^n, n>0$		$\frac{n!}{s^{n+1}}$
e^{-at}	exponential decay	$\frac{1}{s+a}$
$\sin(\omega t)$		$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$		$\frac{s}{s^2+\omega^2}$
te^{-at}		$\frac{1}{(s+a)^2}$
t^2e^{-at}		$\frac{2!}{(s+a)^3}$

27

Laplace transform table (continued)

TIME DOMAIN	FREQUENCY DOMAIN
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\left[B\cos\omega t+\left(\frac{C-aB}{\omega}\right)\sin\omega t\right]$	$\frac{Bs+C}{(s+a)^2+\omega^2}$
$2\|A\|e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{s+\alpha-\beta j}+\frac{A^{\text{complex conjugate}}}{s+\alpha+\beta j}$
$2t\|A\|e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{(s+\alpha-\beta j)^2}+\frac{A^{\text{complex conjugate}}}{(s+\alpha+\beta j)^2}$
$\frac{(c-a)e^{-at}-(c-b)e^{-bt}}{b-a}$	$\frac{s+c}{(s+a)(s+b)}$
$\frac{e^{-at}-e^{-bt}}{b-a}$	$\frac{1}{(s+a)(s+b)}$

28

Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function

$$F(s)=\frac{b_m s^m+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=\frac{N(s)}{D(s)},\quad m<n$$

  – Example: $$\frac{c}{(s+a)(s+b)}=\frac{k_1}{s+a}+\frac{k_2}{s+b}$$

29

Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- Theorems of the Laplace transform
  – Differential theorem

$$L\left[\frac{df}{dt}\right]=sF(s)-f(0^-),$$

$$L\left[\frac{d^n f}{dt^n}\right]=s^n F(s)-s^{n-1}f(0^-)-\cdots-f^{n-1}(0^-),$$

$$\text{where } f(0^-)=\lim_{t\to 0}f(t),\quad t<0$$

30

Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- Theorems of the Laplace transform
  - Differential theorem
  - Integral theorem

$$L\left[\int_0^t f(\tau)d\tau\right]=\frac{F(s)}{s}$$

  - Shifting theorem

$$L[f(t-t_0)u(t-t_0)]=e^{-t_0 s}F(s)$$

31

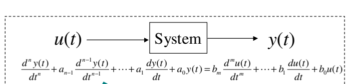
Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- Theorems of the Laplace transform
  - Differential theorem
  - Integral theorem
  - Shifting theorem
  - Theorem of convolution integral

$$L^{-1}[F_1(s)F_2(s)]=\int_0^t f_1(t-\tau)f_2(\tau)d\tau=\int_0^t f_1(\tau)f_2(t-\tau)d\tau$$

32

Transfer functions

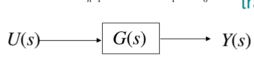


Laplace transform
(with zero initial conditions)

$$(s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0)Y(s)=(b_m s^m+\cdots+b_1 s+b_0)U(s)$$

$$\frac{Y(s)}{U(s)}\equiv G(s)=\frac{b_m s^m+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}$$

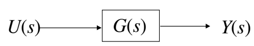
$G(s)$ is called a transfer function



33

Transfer functions

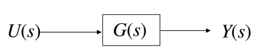
- The transfer function is defined only for a linear time-invariant system (not for nonlinear systems)
- The transfer function between a pair of input and output variables is the ratio of the Laplace transform of the output to the Laplace transform of the input (alternatively, the transfer function between an input variable and an output variable of a system is defined as the Laplace transform of the impulse response)



34

Transfer functions

- The transfer function is defined only for a linear time-invariant system (not for nonlinear systems)
- The transfer function between a pair of input and output variables is the ratio of the Laplace transform of the output to the Laplace transform of the input
- All initial conditions of the system are set to zero
- The transfer function is independent of the input of the system
- The transfer function of a continuous-data system is expressed only as a function of the complex variable s. It is not a function of the real variable, time, or any other variable that is used as the independent variable



35

References

- S. Boyd, Lecture Notes for EE102: Introduction to Signals and Systems, Stanford University.
- D. Heeger, Signals, Linear Systems, and Convolution, http://white.stanford.edu/~heeger/ftp/handouts/convolution.pdf
- D. K. Lindner, Introduction to Signals and Systems, McGraw-Hill, 1999.
- P. J. Nahin. Behind the Laplace transform. IEEE Spectrum, vol. 28, no. 3, pp. 60, March 1991.
- C. L. Phillips and R. D. Harbor. Feedback Control Systems, 4th Edition, Prentice Hall, 2000.
- M. J. Roberts, Fundamentals of Signals and Systems, McGraw-Hill, 2008.
- S. W. Smith, The Scientist and Engineer's Guide to Digital Signal Processing, 2nd Edition, California Technical Publishing, 1999.

36