BioE 2696/ECE 2695: Control Theory in Neuroscience
(3 Credits, Spring 2009)

Optimization and Optimal Control

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Outline

- Brief review of optimization methods
- What is optimal control?
- Why optimal control?
- Approaches to optimal control

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Brief review of optimization methods

- Formulation of optimization problems
  - Objective function
  - Decision variables
  - Constraints

$$\begin{aligned}&\text{minimize} \quad f(x)\\&\text{subject to} \quad g_i(x) \le 0,\ i = 1, \ldots, n\end{aligned}$$

Question: Are all these ingredients necessary?

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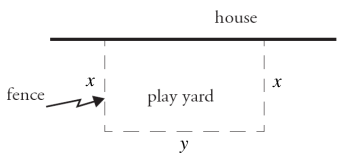
Brief review of optimization methods

- Formulation of optimization problems
- Examples of optimization problems
  - A toy example: A child's rectangular play yard is to be built next to the house. To make the three sides of the play-pen, twenty-four feet of fencing are available. What should be the dimensions of the sides to make a maximum area?



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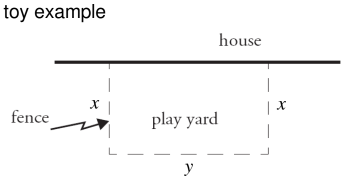
Brief review of optimization methods

- Formulation of optimization problems
- Examples of optimization problems
  - A toy example



$$\text{maximize} \quad xy$$
$$\text{subject to} \quad 2x + y = 24$$

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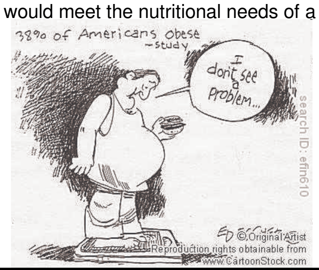
Brief review of optimization methods

- Formulation of optimization problems
- Examples of optimization problems
  - A toy example
  - The diet problem (one of the first modern optimization problems): In the 1930s-1940s, the Army wanted a low cost diet that would meet the nutritional needs of a soldier



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Brief review of optimization methods

- Formulation of optimization problems
- Examples of optimization problems
  - A toy example
  - The diet problem

minimize cost of food
subject to: total calories ≥ minimum requirement,
amount of vitamins ≥ minimum requirement,
amount of minerals ≥ minimum requirement, etc.

9 inequalities, 77 decision variables

Solution: The minimum cost of food is $____ per year!

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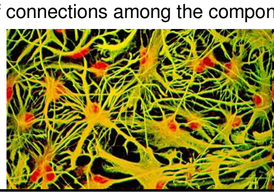
Brief review of optimization methods

- Formulation of optimization problems
- Examples of optimization problems
  - A toy example
  - The diet problem
  - "Save wire" organizing principle: At multiple hierarchical levels—brain, ganglion, individual cell—physical placement of neural components appears consistent with a single, simple goal, i.e., to minimize cost of connections among the components



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Brief review of optimization methods

- Formulation of optimization problems
- Examples of optimization problems
  - A toy example
  - The diet problem
  - "Save wire" organizing principle
  - Optimization in biology
    - Optimization theory not only explains current adaptations of biological systems, but also helps to predict new designs that may yet evolve
    - Biological world may provide solutions to engineering problems: The structures, movements, and behaviors of animals, and their life histories, have been shaped by the optimization processes of evolution or of learning by trial and error

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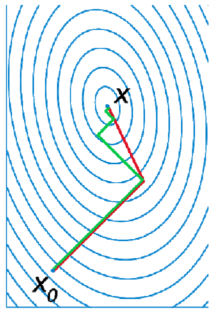
Brief review of optimization methods

- Formulation of optimization problems
- Examples of optimization problems
- Optimization methods
  - Extremum of a smooth function
  - Gradient search
  - Simplex algorithm



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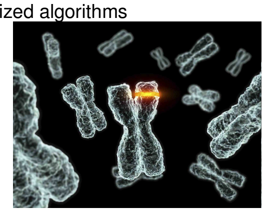
Brief review of optimization methods

- Formulation of optimization problems
- Examples of optimization problems
- Optimization methods
  - Extremum of a smooth function
  - Gradient search
  - Simplex algorithm
  - Lagrangian methods and Lagrange multipliers
  - Randomized algorithms

Genetic algorithm



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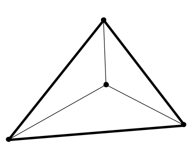
Brief review of optimization methods

- Formulation of optimization problems
- Examples of optimization problems
- Optimization methods
  - Extremum of a smooth function
  - Gradient search
  - Simplex algorithm
  - Lagrangian methods and Lagrange multipliers
  - Randomized algorithms
  - Energy-function based optimization
    - With applications in protein folding problems, Hopfield neural networks, robotic path planning, etc.

Question (**Steiner's problem**): How to find a point inside a triangle that gives the shortest sum of distances to the vertices?



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What is optimal control?

- Definition
  - Optimal control is to find optimal ways to control a dynamic system

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What is optimal control?

- Definition
- Formulation of optimal control problems
  - State-space description of a system
    - The system is modeled as a set of first-order differential equations (representation of the dynamics of an n th-order system using n first-order differential equations)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

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What is optimal control?

- Definition
- Formulation of optimal control problems
  - State-space description of a system
    - The system is modeled as a set of first-order differential equations (representation of the dynamics of an n th-order system using n first-order differential equations)
    - Example: Newton's second law

$$m\frac{d^2y(t)}{dt^2} = u(t)$$

$$x_1(t) = y(t)$$
$$x_2(t) = \frac{dy(t)}{dt} = \frac{dx_1(t)}{dt}$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1/m\end{bmatrix}u(t)$$

$$y(t) = [1 \ \ 0]\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

Question: What are A, B, C, and D for this example?

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What is optimal control?

- Definition
- Formulation of optimal control problems
  - State-space description of a system
  - Objective functions or performance indices
    - Example 1: Suppose that the objective is to control a dynamical system modeled by the equations

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(t_0) = x_0 \\ y &= Cx \end{aligned}$$

on a fixed interval $[t_0, t_f]$ so that the components of the state vector are "small." A suitable performance index to be minimized would be

$$J_1 = \int_{t_0}^{t_f} x^T(t)x(t)dt$$

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What is optimal control?

- Definition
- Formulation of optimal control problems
  - State-space description of a system
  - Objective functions or performance indices
    - Example 1
    - Example 2: If the objective is to control the system so that the components of the output, $y(t)$, are to be small, then we could use the performance index

$$\begin{aligned} J_2 &= \int_{t_0}^{t_f} y^T(t)y(t)dt \\ &= \int_{t_0}^{t_f} x^T(t)C^TCx(t)dt = \int_{t_0}^{t_f} x^T(t)Qx(t)dt \end{aligned}$$

where the weight matrix $Q = C^TC$ is symmetric positive semidefinite

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What is optimal control?

- Definition
- Formulation of optimal control problems
  - State-space description of a system
  - Objective functions or performance indices
    - Example 1
    - Example 2
    - Example 3: If the objective is to control the system so that the components of the iutput, $u(t)$, are to be small, then we could use the performance index

$$J_3 = \int_{t_0}^{t_f} u^T(t)u(t)dt \quad \text{or} \quad J_3 = \int_{t_0}^{t_f} u^T(t)Ru(t)dt$$

where the weight matrix R is symmetric positive definite

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What is optimal control?

- Definition
- Formulation of optimal control problems
  - State-space description of a system
  - Objective functions or performance indices
    - Example 1
    - Example 2
    - Example 3
    - Example 4: If we wish the final state $x(t_f)$ to be as close as possible to 0, then we could use the performance index

$$J_4 = x^T(t_f)Fx(t_f)$$

where F is a symmetric positive definite matrix

What is optimal control?

- Definition
- Formulation of optimal control problems
  - State-space description of a system
  - Objective functions or performance indices
  - LQR (linear quadratic regulator) problem
    - The control aim is to keep the state "small," the control "not too large," and the final state as near to 0 as possible. The resulting performance index is

$$J = x^T(t_f)Fx(t_f) + \int_{t_0}^{t_f} \left(x^T(t)Qx(t) + u^T(t)Ru(t) \right) dt$$

Minimizing the above performance index subject to

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0$$
$$y = Cx$$

is called the LQR problem

What is optimal control?

- Definition
- Formulation of optimal control problems
  - State-space description of a system
  - Objective functions or performance indices
  - LQR (linear quadratic regulator) problem

$$J = x^T(t_f)Fx(t_f) + \int_{t_0}^{t_f} \left(x^T(t)Qx(t) + u^T(t)Ru(t) \right) dt$$

Question: What if the desired state is not 0 but $x_d(t_f)$?

What is optimal control?

- Definition
- Formulation of optimal control problems
- Comparison with conventional optimization problems

Question: What are the decision variables and constraints of an optimal control problem?

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Why optimal control?

- Problems with classical control system design
  - Classical design is a trial-and-error process
  - Classical design is to determine the parameters of an "acceptable" system
  - Classical design is essentially restricted to single-input single-output LTI systems

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Why optimal control?

- Problems with classical control system design
- Why optimal control?
  - Based on state-space description of systems and applicable to control problems involving multi-input multi-output systems and time-varying situations
  - "Optimal" design (in optimal control) v.s. "acceptable" design (in classical control)

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Why optimal control?

- Problems with classical control system design
- Why optimal control?
- Word of caution
  - Optimal control design assumes that the system model is exactly known and that there are no disturbances
  - Lack of intuition in design

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Approaches to optimal control

- Calculus of variations
  - Pontryagin's maximum principle
- Dynamic programming
  - Hamilton-Jacobi-Bellman equation

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Approaches to optimal control

- Calculus of variations
- Dynamic programming
- Linear quadratic regulator
  - For an LTI system described by $\dot{x} = Ax + Bu, \quad x(0) = x_0$ with a quadratic cost function defined as

$$J = \int_0^\infty \left(x^T Q x + u^T R u\right) dt$$

the feedback control law that minimizes the value of the cost is $u = -Kx$ where K is given by $K = R^{-1}B^T P$ and P is found by solving the algebraic Riccati equation (ARE):

$$A^T P + PA + Q - PBR^{-1}B^T P = 0$$

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