Output Feedback

Adapted from Chapter 5 in *Adaptive Control*
(2nd Edition) by K. J. Astrom and B.
Wittenmark, Addison-Wesley, 1995

1. Adaptation of a Feedforward Gain

We now use Lyapunov theory to derive parameter adjustment laws for the problem of adjusting a feedforward gain. We consider the case in which the plant has transfer function $k\hat{G}(s)$, where $\hat{G}(s)$ is known and k is unknown. The desired response is given by the transfer function $k_0\hat{G}(s)$. This problem was discussed previously in Lecture 4 (Example 1). The error is given by

$$e_0 = [k\hat{G}(s)\theta - k_0\hat{G}(s)]r = k\hat{G}(s)(\theta - \theta^*)r$$

where $\theta^* = k_0 / k$. To use Lyapunov theory, we first introduce a state space representation of the transfer function $\hat{G}$. The relation between the parameter θ and the error e_0 can then be written as

$$\begin{aligned} \frac{dx}{dt} &= Ax + B(\theta - \theta^*)r \\ e_0 &= Cx. \end{aligned} \tag{1}$$

If the homogeneous system $\dot{x} = Ax$ is asymptotically stable, there exist positive definite matrices P and Q such that

$$A^T P + PA = -Q\,. \tag{2}$$

Choose the following function as a candidate for a Lyapunov function:

$$v(x,\theta) = \frac{1}{2} g x^T P x + \frac{1}{2}(\theta - \theta^*)^2\,.$$

The time derivative of v along the differential equation, Eqs. (1), is given by

$$\frac{dv}{dt} = \frac{g}{2}\left(\frac{dx^T}{dt}Px + x^T P \frac{dx}{dt}\right) + (\theta - \theta^*)\frac{d\theta}{dt}\,.$$

Using Eqs. (1), we get

$$\begin{aligned} \frac{dv}{dt} &= \frac{g}{2}\Big((Ax + Br(\theta - \theta^*))^T Px + x^T P(Ax + Br(\theta - \theta^*))\Big) + (\theta - \theta^*)\frac{d\theta}{dt} \\ &= -\frac{g}{2}x^T Q x + (\theta - \theta^*)\left(\frac{d\theta}{dt} + grB^T Px\right). \end{aligned}$$

If the parameter adjustment law is chosen to be

$$\frac{d\theta}{dt} = -g\, r\, B^T P x \tag{3}$$

we find that the derivative of the Lyapunov function will be negative as long as $x \neq 0$. The state vector x and the error $e_0 = Cx$ will go to zero as t goes to infinity. Notice, however, that the parameter error will not necessarily go to zero.

Output feedback The result obtained is quite restrictive because it requires that all state variables are known. A parameter adjustment law that uses output feedback can be obtained if the Lyapunov function can be chosen so that

$$B^T P = C$$

where C is the output matrix of the system in Eqs. (1). With this choice of P it follows that

$$B^T P x = C x = e_0$$

and the adjustment rule becomes

$$\frac{d\theta}{dt} = -g\, r\, e_0 \,.$$

The appropriate condition is given by the celebrated Kalman-Yakubovich lemma.

Lemma 1 (Kalman-Yakubovich lemma) (see also Lemma 2.6.2, p. 83, in the textbook by Sastry and Bodson): Let the time-invariant linear system

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$

be completely controllable and completely observable. The transfer function

$$\hat{G}(s) = C(sI - A)^{-1} B$$

is strictly positive real if and only if there exist positive definite matrices P and Q such that

$$A^T P + PA = -Q$$

and

$$B^T P = C \,.$$

Theorem 1 (MRAS using the Lyapunov rule): Consider the problem of adapting a feedforward gain. Assume that the transfer function is strictly positive real. Then the parameter adjustment rule

$$\frac{d\theta}{dt} = -g\, r\, e_0 \tag{4}$$

where g is a positive constant, makes the output error e_0 in Eqs. (1) go to zero.

The control law of Eq. (4) is very similar to the control law obtained by the MIT rule, $d\theta / dt = -gy_m e_0$. However, the adaptation gain g in the control law derived by the MIT rule needs to be small, while there is no such constraint on g in Eq. (4).

2. Output Feedback

We now present an outline for the derivation of an MRAS for adjusting the parameters of a control based on output feedback in a fairly general case. A process with one input and one output is considered. It is assumed that the dynamics are linear and that the control problem is formulated as model-following. The key assumption is that the controller can be parameterized in such a way that the error is linear in the controller parameters. The outline of the derivation of the MRAS is described as follows:

Step 1: Find a controller structure that admits perfect output tracking.

Step 2: Derive an error model of the form

$$\varepsilon = \hat{G}_1(s)\{w^T(t)(\theta - \theta^*)\} \tag{5}$$

where $\hat{G}_1$ is a strictly positive real transfer function, θ is the controller parameters, and θ^* is the process parameters or the nominal value of θ. The right-hand side should be expressed in computable quantities.

Step 3: Use the parameter adjustment law

$$\frac{d\theta}{dt} = -g\, w \varepsilon\,. \tag{6}$$

Notice that the error in Eq. (5) is linear in the parameters, a condition that imposes restrictions on the models and controllers that can be dealt with. A model of the form (5) is typically obtained by algebraic manipulations, filtering, and error augmentation.