Lyapunov Design for General Linear Systems in MRAC

Adapted from Chapter 5 in *Adaptive Control (2nd Edition)* by K. J. Astrom and B. Wittenmark, Addison-Wesley, 1995

We will now show how Lyapunov's theory can be used to derive stable MRASs for general linear systems. The idea is as follows:

1. Find a controller structure.
2. Derive the error equation.
3. Find a Lyapunov function and use it to derive a parameter updating law such that the error will go to zero.

Consider a linear system described by

$$\frac{dx_p}{dt} = A_p x_p + B_p u \,. \tag{1}$$

Assume that it is desired to find a control law so that the response to command signals is given by

$$\frac{dx_m}{dt} = A_m x_m + B_m r \,. \tag{2}$$

A general linear control law for the system given by Eq. (1) is

$$u = Fr - Lx_p \,. \tag{3}$$

The closed-loop system then becomes

$$\frac{dx_p}{dt} = (A_p - B_p L)x_p + B_p Fr = A_c(\theta)x_p + B_c(\theta)r \,. \tag{4}$$

The control law can be parameterized in different ways. All parameters in the matrices L and F may be chosen freely. There may also be constraints among the parameters. The general case can be captured by assuming that the closed-loop system is described by Eq. (4), where matrices A_c and B_c depend on a parameter θ.

<u>Compatibility conditions</u> It is not always possible to find parameters θ such that Eq. (4) is equivalent to Eq. (2). A sufficient condition is that there exists a parameter value θ^* such that

$$\begin{aligned} A_c(\theta^*) &= A_m \\ B_c(\theta^*) &= B_m \end{aligned} \,. \tag{5}$$

This condition for perfect model-following is fairly stringent. When all parameters in the control law can be chosen freely, it implies that

$$A_p - A_m = B_p L$$
$$B_m = B_p F.$$

This means that the columns of matrices $A_p - A_m$ and B_m are linear combinations of the columns of matrix B_p. If these conditions are satisfied and the columns of B_p and B_m are linearly independent, then the matrices L and M are given by

$$L = (B_p{}^T B_p)^{-1} B_p{}^T (A_p - A_m) = (B_m{}^T B_p)^{-1} B_m{}^T (A_p - A_m)$$
$$F = (B_p{}^T B_p)^{-1} B_p{}^T B_m = (B_m{}^T B_p)^{-1} B_m{}^T B_m.$$

<u>The error equation</u> Introduce the error defined as

$$e = x_p - x_m .$$

Subtracting Eq. (2) from Eq. (1) gives

$$\frac{de}{dt} = \frac{dx_p}{dt} - \frac{dx_m}{dt} = A_p x_p + B_p u - A_m x_m - B_m r .$$

Adding and subtracting $A_m x$ from the right-hand side give

$$\begin{aligned}\frac{de}{dt} &= A_m e + (A_p - A_m - B_p L)x + (B_p F - B_m) r \\ &= A_m e + (A_c(\theta) - A_m)x + (B_c(\theta) - B_m) r \\ &= A_m e + (A_c(\theta) - A_c(\theta^*))x + (B_c(\theta) - B_c(\theta^*)) r \\ &= A_m e + \psi(\theta - \theta^*).\end{aligned} \tag{6}$$

To obtain the last equality, it has been assumed that the conditions for exact model-following are satisfied. This is required for θ^* to exist. To derive a parameter adjustment law, we introduce the Lyapunov function

$$v(e,\theta) = \frac{g}{2} e^T P e + \frac{1}{2}(\theta - \theta^*)^T (\theta - \theta^*) \tag{7}$$

where P is a positive definite matrix. The function v is positive definite. To find out whether it can be a Lyapunov function, we calculate its total time derivative

$$\begin{aligned}\frac{dv(e,\theta)}{dt} &= -\frac{g}{2} e^T Q e + g(\theta - \theta^*)^T \psi^T P e + (\theta - \theta^*)^T \frac{d\theta}{dt} \\ &= -\frac{g}{2} e^T Q e + (\theta - \theta^*)^T \left(\frac{d\theta}{dt} + g \psi^T P e \right)\end{aligned}$$

where Q is positive definite and such that

$$A_m^T P + P A_m = -Q .$$

It can be shown that a pair of positive definite matrices P and Q with this property always exist if A_m is stable (see pp. 38-39 in the textbook by Sastry and Bodson).

If the parameter adjustment law is chosen to be

$$\frac{d\theta}{dt} = -g\psi^T Pe \tag{8}$$

we get

$$\frac{dv}{dt} = -\frac{g}{2} e^T Qe$$

The time derivative of the Lyapunov function is negative semidefinite. It can be shown that the error goes to zero. Notice that we have assumed that all states x_p are measurable.