ECE 2695: Adaptive Control (3 Credits, Fall 2008)

Lecture 3: System Identification (I)

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Outline

- Homework 2
- Review of last lecture
- Basic identification methods

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Homework 1

- Problems 1.10, 1.12 (a)-(c), and 2.2 (a)-(b)
- Due on 9/22 (Monday) in class but before the lecture starts

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Review of last lecture

- State-space description of dynamic systems

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0$$

where $x \in \mathbf{R}^n$, $t \geq 0$.

  - Autonomous and non-autonomous systems
  - Equilibrium

> Question: What is the equilibrium of the following system?
> $$\dot{x} = Ax$$

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Review of last lecture

- State-space description of dynamic systems
  - Autonomous and non-autonomous systems
  - Equilibrium
  - Lipschitz condition and consequences
    - Definition: The function f is said to be Lipschitz in x if, for some $h > 0$, there exists $l \geq 0$ such that

$$|f(t,x_1) - f(t,x_2)| \leq l|x_1 - x_2|$$

for all $x_1, x_2 \in B_h$ (closed ball of radius h centered at a point of interest in $\mathbf{R}^n$), $t > 0$. The constant l is called the Lipschitz constant. Globally Lipschitz functions satisfy the above inequality for all $x_1, x_2 \in \mathbf{R}^n$

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Review of last lecture

- State-space description of dynamic systems
  - Autonomous and non-autonomous systems
  - Equilibrium
  - Lipschitz condition and consequences

$$|f(t,x_1) - f(t,x_2)| \leq l|x_1 - x_2|$$

> Remarks:
> (1) The Lipschitz property is by default assumed to be satisfied uniformly, i.e., l does not depend on t.
> (2) If f is Lipschitz in x, then it is continuous in x.
> (3) If f has continuous and bounded partial derivatives in x, then it is Lipschitz.

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Review of last lecture

- State-space description of dynamic systems
  - Autonomous and non-autonomous systems
  - Equilibrium
  - Lipschitz condition and consequences
    - Definition

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0 \qquad (*)$$

    - A sufficient condition for the existence and uniqueness of the solutions of (*) on some time interval (for as long as $x \in B_h$): f locally bounded and locally Lipschitz in x

Review of last lecture

- State-space description of dynamic systems
  - Autonomous and non-autonomous systems
  - Equilibrium
  - Lipschitz condition and consequences
    - Definition
    - A sufficient condition for the existence and uniqueness of the solutions
    - Examples

What if x is a vector?

a) $f(t,x) = \frac{x}{1+x^2}$ b) $f(t,x) = \frac{e^t}{1+e^t}\, x$

c) $f(t,x) = A\,x$ d) $f(t,x) = e^t\, x$

e) $f(t,x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ f) $f(t,x) = \sqrt{x}$

Review of last lecture

- State-space description of dynamic systems
- Stability of dynamic systems
  - Stability in LTI systems

The zero-input response of the following system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

or the equation

$$\dot{x} = Ax$$

is marginally stable or stable in the sense of Lyapunov if every finite initial state excites a bounded response. It is asymptotically stable if every finite initial state excites a bounded response, which in addition, approaches $\mathbf{0}$ as t approaches ∞

Review of last lecture

- State-space description of dynamic systems
- Stability of dynamic systems
  - Stability in LTI systems

If A has distinct eigenvalues, then the equation $\dot{x} = Ax$ is marginally stable if and only if all eigenvalues of A have zero or negative real parts.

The equation $\dot{x} = Ax$ is marginally stable if and only if all eigenvalues of A have zero or negative real parts and those with zero real parts are simple roots of the minimal polynomial of A.

The equation $\dot{x} = Ax$ is asymptotically stable if and only all eigenvalues of A have negative real parts.

Review of last lecture

- State-space description of dynamic systems
- Stability of dynamic systems
  - Stability in LTI systems
    - Examples

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad A_5 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Review of last lecture

- State-space description of dynamic systems
- Stability of dynamic systems
  - Stability in LTI systems

$$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0 \quad (*)$$

  - Stability in the sense of Lyapunov
    - The point $x = 0$ is called a stable equilibrium point of (*), if, for all $t_0 \geq 0$ and $\varepsilon > 0$, there exists $\delta(t_0, \varepsilon)$ such that

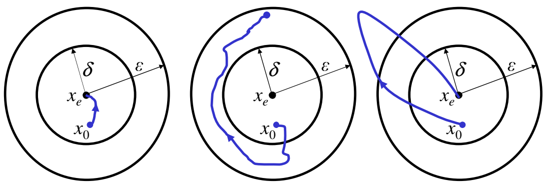
$$|x_0| < \delta(t_0, \varepsilon) \Rightarrow |x(t)| < \varepsilon \text{ for all } t \geq t_0$$

where $x(t)$ is the solution of (*) starting from x_0 at t_0

Question: How to generalize the above definition to a nonzero equilibrium?

Review of last lecture

- State-space description of dynamic systems
- Stability of dynamic systems
  - Stability in LTI systems
  - Stability in the sense of Lyapunov



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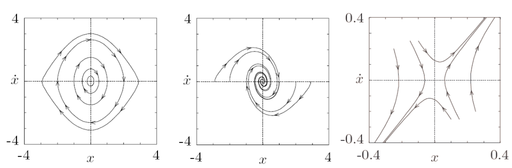
Review of last lecture

- State-space description of dynamic systems
- Stability of dynamic systems
  - Stability in LTI systems
  - Stability in the sense of Lyapunov
    - Examples



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Review of last lecture

- State-space description of dynamic systems
- Stability of dynamic systems
  - Stability in LTI systems
  - Stability in the sense of Lyapunov
  - Uniform stability
  - Asymptotic stability
  - Uniform asymptotic stability
  - Global asymptotic stability
  - Exponential stability
  - Global exponential stability

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Review of last lecture

- State-space description of dynamic systems
- Stability of dynamic systems
- Lyapunov stability theory
  - Basic idea: If some "measure of the energy" associated with a system is decreasing, then the system will converge to its equilibrium

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Review of last lecture

- State-space description of dynamic systems
- Stability of dynamic systems
- Lyapunov stability theory
  - Basic idea
  - A few concepts
    - Class K functions: A function $\alpha(\cdot)$: $\mathbf{R}_+ \to \mathbf{R}_+$ belongs to class K (denoted $\alpha(\cdot) \in K$), if it is continuous, strictly increasing, and $\alpha(0) = 0$

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Review of last lecture

- State-space description of dynamic systems
- Stability of dynamic systems
- Lyapunov stability theory
  - Basic idea
  - A few concepts
    - Class K functions
    - Locally positive definite functions (l.p.d.f.): A continuous function $v(t, x)$: $\mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}_+$ is called a locally positive definite function (l.p.d.f.) if, for some $h > 0$, and some $\alpha(\cdot) \in K$

$$v(t,0) = 0 \quad \text{and} \quad v(t,x) \geq \alpha(|x|) \quad \text{for all} \quad x \in B_h, t \geq 0$$

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Review of last lecture

- State-space description of dynamic systems
- Stability of dynamic systems
- Lyapunov stability theory
  - Basic idea
  - A few concepts
    - Class K functions
    - Locally positive definite functions (l.p.d.f.)
    - Positive definite functions (p.d.f.): A continuous function $v(t, x)$: $\mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}_+$ is called a positive definite function (p.d.f.) if, for some $\alpha(\cdot) \in K$

$$v(t,0)=0 \quad \text{and} \quad v(t,x) \geq \alpha(|x|) \quad \text{for all } x \in \mathbf{R}^n, t \geq 0$$

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Review of last lecture

- State-space description of dynamic systems
- Stability of dynamic systems
- Lyapunov stability theory
  - Basic idea
  - A few concepts
    - Class K functions
    - Locally positive definite functions (l.p.d.f.)
    - Positive definite functions (p.d.f.)

Question: Are these functions p.d.f.?

$$v(t,x) = |x|^2 \qquad v(t,x) = x^T P x, \text{ with } P > 0$$

$$v(t,x) = (t+1)|x|^2 \qquad v(t,x) = e^{-t}|x|^2$$

$$v(t,x) = \sin^2(|x|^2)$$

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Review of last lecture

- State-space description of dynamic systems
- Stability of dynamic systems
- Lyapunov stability theory
  - Basic idea
  - A few concepts
    - Class K functions
    - Locally positive definite functions (l.p.d.f.)
    - Positive definite functions (p.d.f.)
    - Decrescent functions: The function $v(t, x)$ is called decrescent, if there exists a function $\beta(\cdot) \in K$, such that

$$v(t,x) \leq \beta(|x|) \quad \text{for all } x \in B_h, t \geq 0$$

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Review of last lecture

- State-space description of dynamic systems
- Stability of dynamic systems
- Lyapunov stability theory
  - Basic idea
  - A few concepts
    - Class K functions
    - Locally positive definite functions (l.p.d.f.)
    - Positive definite functions (p.d.f.)
    - Decrescent functions

Question: Are these functions decrescent functions?

$v(t,x) = |x|^2$ $\qquad v(t,x) = x^T P x$, with $P > 0$

$v(t,x) = (t+1)|x|^2$ $\qquad v(t,x) = e^{-t}|x|^2$

$v(t,x) = \sin^2(|x|^2)$

Review of last lecture

- State-space description of dynamic systems
- Stability of dynamic systems
- Lyapunov stability theory
  - Basic idea
  - A few concepts
  - Basic theorem of Lyapunov
    - Let $v(t, x)$ be continuously differentiable. Then

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0 \qquad (*)$$

$$\frac{dv(t,x)}{dt} = \frac{\partial v(t,x)}{\partial t} + \frac{\partial v(t,x)}{\partial x} f(t,x)$$

Conditions on $v(t, x)$	Conditions on $-dv(t, x)/dt$	Conclusions
l.p.d.f.	≥ 0 locally	stable
l.p.d.f., decrescent	≥ 0 locally	uniformly stable
l.p.d.f.	l.p.d.f.	asymptotically stable
l.p.d.f., decrescent	l.p.d.f.	uniformly asymptotically
p.d.f., decrescent	p.d.f.	globally u.a.s.

Review of last lecture

- State-space description of dynamic systems
- Stability of dynamic systems
- Lyapunov stability theory
  - Basic idea
  - A few concepts
  - Basic theorem of Lyapunov
    - Examples

$$\dot{x}_1 = x_1(x_1^2 + x_2^2 - 1) - x_2$$

$$\dot{x}_2 = x_1 + x_2(x_1^2 + x_2^2 - 1)$$

$$v(x_1, x_2) = x_1^2 + x_2^2$$

Review of last lecture

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  - Basic idea
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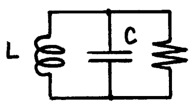
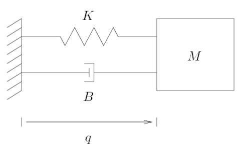
  - Basic theorem of Lyapunov
    - Examples





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Review of last lecture

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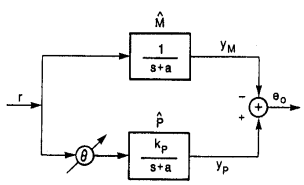
- Lyapunov stability theory
  - Basic idea
  - A few concepts
  - Basic theorem of Lyapunov
    - Examples

MRAC using the MIT rule and the Lyapunov redesign



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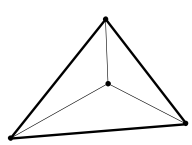
Review of last lecture

- State-space description of dynamic systems
- Stability of dynamic systems
- Lyapunov stability theory
  - Basic idea
  - A few concepts
  - Basic theorem of Lyapunov
    - Examples

Question (**Steiner's problem**): How to find a point inside a triangle that gives the shortest sum of distances to the vertices?

Energy-function based optimization algorithms (with applications in protein folding problems, Hopfield neural networks, robotic path planning, etc.)



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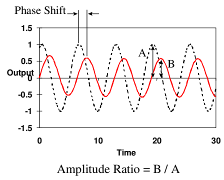
Basic identification methods

- Frequency domain approach
  - Frequency response: steady-state response of systems to sinusoidal inputs

The figure compares the output response of a system (red solid line) with a sinusoidal input (black dashed line)

Both the magnitude and the phase shift of a system will change with the frequency of the input into the system



Basic identification methods

- Frequency domain approach
  - Frequency response: steady-state response of systems to sinusoidal inputs
  - Frequency response function
    - Given a system with transfer function $\hat{P}(s)$, its frequency response function is $\hat{P}(j\omega)$
    - The steady-state gain of a system for a sinusoidal input $\sin(\omega_0 t)$ is the magnitude of the transfer function evaluation at $s = j\omega_0$, and the phase shift of the output sinusoid relative to the input sinusoid is the angle of $\hat{P}(j\omega_0)$

Basic identification methods

- Frequency domain approach
  - Frequency response: steady-state response of systems to sinusoidal inputs
  - Frequency response function
  - Example

$$\frac{\hat{y}_p(s)}{\hat{r}(s)} = \hat{P}(s) = \frac{k_p}{s + a_p}$$

Basic identification methods

- Frequency domain approach
  - Frequency response: steady-state response of systems to sinusoidal inputs
  - Frequency response function
  - Example

Question: How about identification of a system of higher order?

$$\frac{\hat{y}_p(s)}{\hat{r}(s)} = \hat{P}(s) = \frac{\alpha_n s^{n-1} + \cdots + \alpha_1}{s^n + \beta_n s^{n-1} + \cdots + \beta_1}$$

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Basic identification methods

- Frequency domain approach
- Time domain approach
  - Example

$$\frac{\hat{y}_p(s)}{\hat{r}(s)} = \hat{P}(s) = \frac{k_p}{s + a_p}$$

Question: What is the differential equation for the above frequency-domain description?

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Basic identification methods

- Frequency domain approach
- Time domain approach
  - Example

$$\frac{\hat{y}_p(s)}{\hat{r}(s)} = \hat{P}(s) = \frac{k_p}{s + a_p}$$

$$\downarrow$$

$$\dot{y}_p(t) = -a_p y_p(t) + k_p r(t)$$

Question: If we have measurement of $r(t)$ and $dy_p(t)/dt$ at t_1 and t_2, how can we estimate a_p and k_p?

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Basic identification methods

- Frequency domain approach
- Time domain approach
  - Example

$$\frac{\hat{y}_p(s)}{\hat{r}(s)} = \hat{P}(s) = \frac{k_p}{s+a_p}$$

$$\dot{y}_p(t) = -a_p y_p(t) + k_p r(t)$$

$$\begin{bmatrix} -a_p \\ k_p \end{bmatrix} = \begin{bmatrix} y_p(t_1) & r(t_1) \\ y_p(t_2) & r(t_2) \end{bmatrix}^{-1} \begin{bmatrix} \dot{y}_p(t_1) \\ \dot{y}_p(t_2) \end{bmatrix}$$

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Basic identification methods

- Frequency domain approach
- Time domain approach
  - Example

$$\frac{\hat{y}_p(s)}{\hat{r}(s)} = \hat{P}(s) = \frac{k_p}{s+a_p} \Rightarrow \dot{y}_p(t) = -a_p y_p(t) + k_p r(t)$$

$$\begin{bmatrix} -a_p \\ k_p \end{bmatrix} = \begin{bmatrix} y_p(t_1) & r(t_1) \\ y_p(t_2) & r(t_2) \end{bmatrix}^{-1} \begin{bmatrix} \dot{y}_p(t_1) \\ \dot{y}_p(t_2) \end{bmatrix}$$

Question: Why we avoid the measurement of $dy_p(t)/dt$ and how to do that?

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Basic identification methods

- Frequency domain approach
- Time domain approach
  - Example

$$\frac{\hat{y}_p(s)}{\hat{r}(s)} = \hat{P}(s) = \frac{k_p}{s+a_p} \Rightarrow \frac{s+a_p}{s+\lambda}\hat{y}_p = \frac{k_p}{s+\lambda}\hat{r}$$

Define $\hat{w}^{(1)} = \frac{1}{s+\lambda}\hat{r}$, $\hat{w}^{(2)} = \frac{1}{s+\lambda}\hat{y}_p$

$$y_p(t) = k_p w^{(1)}(t) + (\lambda - a_p) w^{(2)}(t)$$
$$\dot{w}^{(1)} = -\lambda w^{(1)} + r$$
$$\dot{w}^{(2)} = -\lambda w^{(2)} + y_p$$

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Basic identification methods

- Frequency domain approach
- Time domain approach
  - Example

$$\frac{\hat{y}_p(s)}{\hat{r}(s)} = \hat{P}(s) = \frac{k_p}{s+a_p}$$

$$\begin{aligned} y_p(t) &= k_p w^{(1)}(t) + (\lambda - a_p) w^{(2)}(t) \\ \dot{w}^{(1)} &= -\lambda w^{(1)} + r \\ \dot{w}^{(2)} &= -\lambda w^{(2)} + y_p \end{aligned}$$

Define $\theta^* := \begin{bmatrix} k_p \\ \lambda - a_p \end{bmatrix}$

$$w(t) := \begin{bmatrix} w^{(1)}(t) \\ w^{(2)}(t) \end{bmatrix}$$

$$y_p(t) = \theta^{*T} w(t) = w^T(t)\theta^*$$

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Basic identification methods

- Frequency domain approach
- Time domain approach
  - Example

$$y_p(t) = \theta^{*T} w(t) = w^T(t)\theta^*$$

Nominal identifier parameter

$$e_1(t) = \theta^T w(t) - y_p(t) = \left[\theta^T - \theta^{*T}\right] w(t)$$

Identification error

Adaptive identifier parameter

Linear error equation

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Basic identification methods

- Frequency domain approach
- Time domain approach
  - Example
  - Gradient algorithm

$$e_1(t) = \theta^T w(t) - y_p(t) = \left[\theta^T - \theta^{*T}\right] w(t)$$

Objective: minimize $e_1^2(t)$

Gradient: $\frac{\partial}{\partial \theta}\left(e_1^2\right) = 2e_1 \frac{\partial}{\partial \theta}\left(e_1\right) = 2e_1 w$

It is a vector!

Parameter update law: $\frac{d\theta}{dt} = -g e_1 w$

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Basic identification methods

- Frequency domain approach
- Time domain approach
  - Example
  - Gradient algorithm
    - Advantages of using the gradient algorithm over using the following calculation

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} w_1(t_1) & w_2(t_1) \\ w_1(t_2) & w_2(t_2) \end{bmatrix}^{-1} \begin{bmatrix} y_p(t_1) \\ y_p(t_2) \end{bmatrix}$$

Basic identification methods

- Frequency domain approach
- Time domain approach
  - Example
  - Gradient algorithm
  - Least-squares algorithm

Objective: minimize the integral-squared-error (ISE)

$$\text{ISE} = \int_0^t \left[\theta^T(\tau)w(\tau) - y_p(\tau)\right]^2 d\tau$$

$$\theta_{LS}(t) = \left\{ \int_0^t w(\tau)\, w^T(\tau)\, d\tau \right\}^{-1} \left\{ \int_0^t w(\tau)\, y_p(\tau)\, d\tau \right\}$$

Least-squares estimate

Basic identification methods

- Frequency domain approach
- Time domain approach
  - Example
  - Gradient algorithm
  - Least-squares algorithm

Recursive formulation

$$\dot{\theta}(t) = -P(t)w(t)\left[\theta^T(t)w(t) - y_p(t)\right] \quad \theta(0) = \theta_0$$

$$\dot{P}(t) = -P(t)w(t)w^T(t)P(t) \quad P(0) = P^T(0) = P_0$$

Remark: It can be shown that $\theta(t)$ converges asymptotically to θ^* if $\int_0^t w(\tau)w^T(\tau)d\tau$ is unbounded as $t \to \infty$.

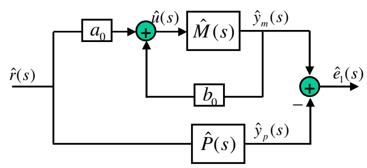
Basic identification methods

- Frequency domain approach
- Time domain approach
  - Example
  - Gradient algorithm
  - Least-squares algorithm
  - Model reference identification

Reference model

$$\frac{\hat{y}_p(s)}{\hat{r}(s)} = \hat{P}(s) = \frac{k_p}{s+a_p}$$

$$\frac{\hat{y}_m(s)}{\hat{u}(s)} = \hat{M}(s) = \frac{k_m}{s+a_m}$$



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