ECE 2695: Adaptive Control (3 Credits, Fall 2008)

Lecture 2: Mathematical Description of Systems and Lyapunov Stability Theory

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Outline

- Homework 1
- Review of last lecture
- Mathematical description of systems
- Stability of dynamic systems

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Homework 1

- Problems 1.9 and 1.11
- Due on 9/15 (Monday) in class but before the lecture starts

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Review of last lecture

- Why adaptive control?
- What is adaptive control?
  - Comparison with ordinary feedback control
  - Comparison with robust control
  - Three time constants
- Approaches to adaptive control

Review of last lecture

- Why adaptive control?
- What is adaptive control?
- Approaches to adaptive control
  - Gain scheduling
  - Model reference adaptive control (MRAC)
    - Series high-gain scheme
    - Parallel scheme

Review of last lecture

- Why adaptive control?
- What is adaptive control?
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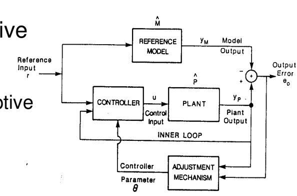


Gradient update

$$\frac{d\theta}{dt} = -g\frac{\partial}{\partial\theta}\left(e_0^2(\theta)\right)$$

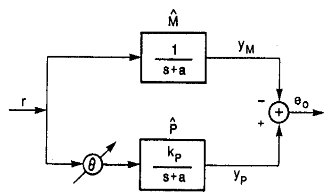
$$= -2ge_0(\theta)\frac{\partial}{\partial\theta}\left(e_0(\theta)\right) = -2ge_0(\theta)\frac{\partial}{\partial\theta}\left(y_p(\theta)\right)$$

Adaptation gain

Sensitivity function

Review of last lecture

- Why adaptive control?
- What is adaptive control?
- Approaches to adaptive control
  - Gain scheduling
  - Model reference adaptive control (MRAC)



Review of last lecture

- Why adaptive control?
- What is adaptive control?
- Approaches to adaptive control
  - Gain scheduling
  - Model reference adaptive control (MRAC)
  - Self tuning regulators
    - Direct and indirect adaptive control
  - Stochastic control approach
    - Dual control

Mathematical description of systems

- Linear time-invariant systems
  - Transfer function

$$u(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

Models in classical control ≈
differential equations of dynamical systems

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_m\frac{d^m u(t)}{dt^m} + \cdots + b_1\frac{du(t)}{dt} + b_0 u(t)$$

Mathematical description of systems

- Linear time-invariant systems
  - Transfer function

$$u(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_m\frac{d^m u(t)}{dt^m} + \cdots + b_1\frac{du(t)}{dt} + b_0 u(t)$$

Laplace transform
(with zero initial conditions)

$$\left(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right)Y(s) = \left(b_m s^m + \cdots + b_1 s + b_0\right)U(s)$$

$$\frac{Y(s)}{U(s)} \equiv G(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$G(s)$ is called a transfer function

$$U(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

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Mathematical description of systems

- Linear time-invariant systems
  - Transfer function

Exercise: Find out the relation between u and y:

$$I\frac{d^2 y(t)}{dt^2} + B\frac{dy(t)}{dt} + Ky(t) = \frac{dx(t)}{dt} + x(t)$$

$$M\frac{d^2 x(t)}{dt^2} + Dx(t) = u(t)$$

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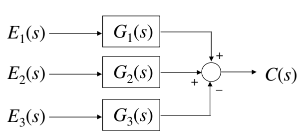
Mathematical description of systems

- Linear time-invariant systems
  - Transfer function

The transfer function relationship $C(s) = G(s)\,E(s)$ can be graphically denoted through a block diagram

$$E(s) \longrightarrow \boxed{G(s)} \longrightarrow C(s)$$

Summing junction in a block diagram



$C(s) = G_1(s)\,E_1(s) + G_2(s)\,E_2(s) - G_3(s)\,E_3(s)$

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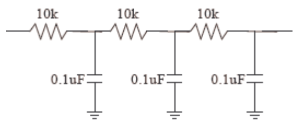
Mathematical description of systems

- Linear time-invariant systems
  - Transfer function

Transfer function of a cascaded system



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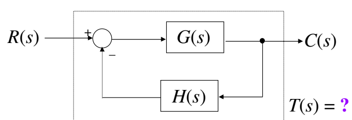
Mathematical description of systems

- Linear time-invariant systems
  - Transfer function



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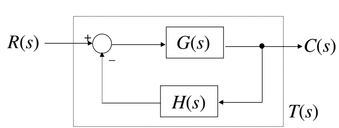
Mathematical description of systems

- Linear time-invariant systems
  - Transfer function



$$T(s) = \frac{\text{Gain of the feedforward path}}{1 - \text{Gain of the loop}}$$

$$= \frac{G(s)}{1-(-1)G(s)H(s)} = \frac{G(s)}{1+G(s)H(s)}$$

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Mathematical description of systems

- Linear time-invariant systems
  - Transfer function

Examples of transfer functions

$\boxed{k}$ $\boxed{s}$ $\boxed{\frac{1}{s}}$ $\boxed{\frac{1}{s+p}}$ $\boxed{\frac{K}{\tau s+1}}$ $\boxed{\frac{s+z}{s+p}}$

$\boxed{\frac{\omega_n^2}{s^2+2\varsigma\omega_n s+\omega_n^2}}$ $\boxed{K_p+\frac{K_I}{s}+\frac{K_D s}{\tau s+1}}$ $\boxed{e^{-Ts}}$

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Mathematical description of systems

- Linear time-invariant systems
  - Transfer function
  - State-space description

The system is modeled as a set of first-order differential equations (representation of the dynamics of an n th-order system using n first-order differential equations)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

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Mathematical description of systems

- Linear time-invariant systems
  - Transfer function
  - State-space description

Advantages of using state-space description:

State variable form is a convenient way to work with complex dynamics; matrix format is easy to use on computers

Transfer functions only deal with input/output behavior, while state-space form provides easy access to the internal features and response of the system

State-space approach is great for MIMO (multi-input multi-output) system, which are very hard to work with using transfer functions

State space can be used to study more general models: The ODEs do not have to be linear

State space introduces the ideas of geometry into differential equations (remember the phase plane in physics?)

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Mathematical description of systems

- Linear time-invariant systems
  - Transfer function
  - State-space description

Question: How to derive the transfer function or transfer-function matrix from a state-space equation?

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

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Mathematical description of systems

- Linear time-invariant systems
  - Transfer function
  - State-space description

Deriving transfer-function matrix from state-space equation

$$\boxed{\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{aligned}} \xrightarrow{\text{Laplace transform}} \boxed{\begin{aligned}s\hat{x}(s) - x(0) &= A\hat{x}(s) + B\hat{u}(s)\\ \hat{y}(s) &= C\hat{x}(s) + D\hat{u}(s)\end{aligned}}$$

$$\boxed{\begin{aligned}\hat{x}(s) &= (sI - A)^{-1}x(0) + (sI - A)^{-1}B\hat{u}(s)\\ \hat{y}(s) &= C(sI - A)^{-1}x(0) + C(sI - A)^{-1}B\hat{u}(s) + D\hat{u}(s)\end{aligned}}$$

For zero initial state

$$\boxed{\hat{y}(s) = [C(sI - A)^{-1}B + D]\,\hat{u}(s)} \rightarrow \boxed{\hat{G}(s) = C(sI - A)^{-1}B + D}$$

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Mathematical description of systems

- Linear time-invariant systems
  - Transfer function
  - State-space description

Question: How to find the state-space equation for a transfer function?

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Mathematical description of systems

- Linear time-invariant systems
  - Transfer function
  - State-space description

Finding the state-space equation for a transfer function

$\hat{g}(s) = \hat{g}(\infty) + \hat{g}_{sp}(s)$,
where $\hat{g}(s)$ is a proper rational function
and $\hat{g}_{sp}(s)$ is the strictly proper part of $\hat{g}(s)$

$$\hat{g}_{sp}(s) = \frac{n(s)}{d(s)} = \frac{n_1 s^{r-1} + n_2 s^{r-2} + \cdots + n_{r-1} s + n_r}{s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1} s + \alpha_r}$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{r-1} & -\alpha_r \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = [n_1 \ \ n_2 \ \ \cdots \ \ n_{r-1} \ \ n_r]\boldsymbol{x} + \hat{g}(\infty) u$$

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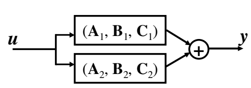
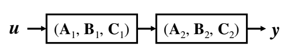
Mathematical description of systems

- Linear time-invariant systems
  - Transfer function
  - State-space description

Exercise: Find out the state-space descriptions for the below connected systems:





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Mathematical description of systems

- Linear time-invariant systems
- Nonlinear systems
  - Differential equations

$$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0$$

where $x \in \mathbf{R}$, $t \geq 0$.

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Mathematical description of systems

- Linear time-invariant systems
- Nonlinear systems
  - Differential equations
    - Autonomous or time-invariant

$$\boxed{\frac{dx}{dt} = f(x), \quad x(t_0) = x_0}$$

    - Non-autonomous or time-varying

$$\boxed{\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0}$$

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Mathematical description of systems

- Linear time-invariant systems
- Nonlinear systems
  - Differential equations $\boxed{\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0}$
  - Lipschitz condition and consequences
    - Definition: The function f is said to Lipschitz in x if, for some $h > 0$, there exists $l \geq 0$ such that

$$|f(t, x_1) - f(t, x_2)| \leq l|x_1 - x_2|$$

for all $x_1, x_2 \in B_h$ (closed ball of radius h centered at 0 in $\mathbf{R}^n$), $t > 0$. The constant l is called the Lipschitz constant. Globally Lipschitz functions satisfy the above inequality for all $x_1, x_2 \in \mathbf{R}^n$

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Mathematical description of systems

- Linear time-invariant systems
- Nonlinear systems
  - Differential equations
  - Lipschitz condition and consequences
    - Definition

$$\boxed{\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0 \qquad (*)}$$

    - A sufficient condition for the existence and uniqueness of the solutions of (*): f locally bounded and locally Lipschitz in x

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Mathematical description of systems

- Linear time-invariant systems
- Nonlinear systems
  - Differential equations
  - Lipschitz condition and consequences
    - Definition

$$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0 \qquad (*)$$

    - A sufficient condition for the existence and uniqueness of the solutions of (*)
    - An example

$$\frac{dx}{dt} = \sqrt{|x|}, \quad x(0) = 0$$

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Mathematical description of systems

- Linear time-invariant systems
- Nonlinear systems
  - Differential equations
  - Lipschitz condition and consequences

$$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0 \qquad (*)$$

  - Equilibrium point: x is called an equilibrium point of the above system if $f(t, x) = 0$ for all $t \geq 0$

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Stability of dynamic systems

- Stability definitions
  - BIBO stability
  - Internal stability of an LTI system

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Stability of dynamic systems

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0 \quad (*)$$

- Stability definitions
  - BIBO stability
  - Internal stability of an LTI system
  - Stability in the sense of Lyapunov
    - The point $x = 0$ is called a stable equilibrium point of (*), if, for all $t_0 \geq 0$ and $\varepsilon > 0$, there exists $\delta(t_0, \varepsilon)$ such that

$$|x_0| < \delta(t_0, \varepsilon) \Rightarrow |x(t)| < \varepsilon \quad \text{for all } t \geq t_0$$

where $x(t)$ is the solution of (*) starting from x_0 at t_0

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Stability of dynamic systems

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0 \quad (*)$$

- Stability definitions
  - BIBO stability
  - Internal stability of an LTI system
  - Stability in the sense of Lyapunov
    - The point $x = 0$ is called a stable equilibrium point of (*), if, for all $t_0 \geq 0$ and $\varepsilon > 0$, there exists $\delta(t_0, \varepsilon)$ such that

$$\boxed{|x_0| < \delta(t_0, \varepsilon) \Rightarrow |x(t)| < \varepsilon \quad \text{for all } t \geq t_0}$$

where $x(t)$ is the solution of (*) starting from x_0 at t_0

  - Uniform stability
    - The point $x = 0$ is called a uniformly stable equilibrium point of (*), if, for all $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that

$$|x_0| < \delta(\varepsilon) \Rightarrow |x(t)| < \varepsilon \quad \text{for all } t \geq t_0$$

where $x(t)$ is the solution of (*) starting from x_0 at t_0

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Stability of dynamic systems

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0 \quad (*)$$

- Stability definitions
  - BIBO stability
  - Internal stability of an LTI system
  - Stability in the sense of Lyapunov
  - Uniform stability
  - Asymptotic stability
    - The point $x = 0$ is called an asymptotically stable equilibrium point of (*), if

(a) $x = 0$ is a stable equilibrium point of (*)

(b) $x = 0$ is attractive, that is, for all for all $t_0 \geq 0$, there exists $\delta(t_0)$ such that

$$|x_0| < \delta \Rightarrow \lim_{t \to \infty} |x(t)| = 0$$

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Stability of dynamic systems

- Stability definitions

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0 \quad (*)$$

  - BIBO stability
  - Internal stability of an LTI system
  - Stability in the sense of Lyapunov
  - Uniform stability
  - Asymptotic stability
  - Uniform asymptotic stability
    - The point $x = 0$ is called a uniformly asymptotically stable equilibrium point of (*), if

      (a) $x = 0$ is a uniformly stable equilibrium point of (*)

      (b) the trajectory $x(t)$ converge to 0 uniformly in t_0

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Stability of dynamic systems

- Stability definitions

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0 \quad (*)$$

  - BIBO stability
  - Internal stability of an LTI system
  - Stability in the sense of Lyapunov
  - Uniform stability
  - Asymptotic stability
  - Uniform asymptotic stability
  - Global asymptotic stability
    - The point $x = 0$ is called a globally asymptotically stable equilibrium point of (*), if it is asymptotically stable and the trajectory $x(t)$ converge to 0 for all x_0

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Stability of dynamic systems

- Stability definitions

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0 \quad (*)$$

  - BIBO stability
  - Internal stability of an LTI system
  - Stability in the sense of Lyapunov
  - Uniform stability
  - Asymptotic stability
  - Uniform asymptotic stability
  - Global asymptotic stability
  - Exponential stability
    - The point $x = 0$ is called an exponential stable equilibrium point of (*), if there exist $m, \alpha > 0$ such that the solution $x(t)$ satisfies

$$|x(t)| \le m e^{-\alpha(t-t_0)} |x_0|$$

      for all $x_0 \in B_h$, $t \ge t_0 \ge 0$. The constant α is called the rate of convergence

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Stability of dynamic systems

- Stability definitions
  - BIBO stability
  - Internal stability of an LTI system
  - Stability in the sense of Lyapunov
  - Uniform stability
  - Asymptotic stability
  - Uniform asymptotic stability
  - Global asymptotic stability
  - Exponential stability
  - Global exponential stability

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0 \qquad (*)$$

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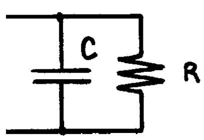
Stability of dynamic systems

- Stability definitions
- Lyapunov stability theory
  - Basic idea: If some "measure of the energy" associated with a system is decreasing, then the system will converge to its equilibrium



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Stability of dynamic systems

- Stability definitions
- Lyapunov stability theory
  - Basic idea
  - A few concepts
    - Class K functions: A function $\alpha(\cdot)$: $\mathbf{R}_+ \to \mathbf{R}_+$ belongs to class K (denoted $\alpha(\cdot) \in K$), if it is continuous, strictly increasing, and $\alpha(0) = 0$

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Stability of dynamic systems

- Stability definitions
- Lyapunov stability theory
  - Basic idea
  - A few concepts
    - Class K functions
    - Locally positive definite functions (l.p.d.f.): A continuous function $v(t, x)$: $\mathbf{R}_+ \times \mathbf{R}^n \rightarrow \mathbf{R}_+$ is called a locally positive definite function (l.p.d.f.) if, for some $h > 0$, and some $\alpha(\cdot) \in K$

$$v(t,0) = 0 \quad \text{and} \quad v(t,x) \geq \alpha(|x|) \quad \text{for all } x \in B_h, t \geq 0$$

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Stability of dynamic systems

- Stability definitions
- Lyapunov stability theory
  - Basic idea
  - A few concepts
    - Class K functions
    - Locally positive definite functions (l.p.d.f.)
    - Positive definite functions (p.d.f.): A continuous function $v(t, x)$: $\mathbf{R}_+ \times \mathbf{R}^n \rightarrow \mathbf{R}_+$ is called a positive definite function (p.d.f.) if, for some $\alpha(\cdot) \in K$

$$v(t,0) = 0 \quad \text{and} \quad v(t,x) \geq \alpha(|x|) \quad \text{for all } x \in \mathbf{R}^n, t \geq 0$$

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Stability of dynamic systems

- Stability definitions
- Lyapunov stability theory
  - Basic idea
  - A few concepts
    - Class K functions
    - Locally positive definite functions (l.p.d.f.)
    - Positive definite functions (p.d.f.)
    - Decrescent functions: The function $v(t, x)$ is called decrescent, if there exists a function $\beta(\cdot) \in K$, such that

$$v(t,x) \leq \beta(|x|) \quad \text{for all } x \in B_h, t \geq 0$$

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Stability of dynamic systems

- Stability definitions
- Lyapunov stability theory
  - Basic idea
  - A few concepts
  - Basic theorem of Lyapunov
    - Let $v(t, x)$ be continuously differentiable. Then

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0 \qquad (*)$$

$$\frac{dv(t,x)}{dt} = \frac{\partial v(t,x)}{\partial t} + \frac{\partial v(t,x)}{\partial x} f(t,x)$$

Conditions on $v(t, x)$	Conditions on $-dv(t, x)/dt$	Conclusions
l.p.d.f.	≥ 0 locally	stable
l.p.d.f., decrescent	≥ 0 locally	uniformly stable
l.p.d.f.	l.p.d.f.	asymptotically stable
l.p.d.f., decrescent	l.p.d.f.	uniformly asymptotically
p.d.f., decrescent	p.d.f.	globally u.a.s.

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References

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- S. Sastry and M. Bodson, Adaptive Control: Stability, Convergence, and Robustness, Prentice-Hall, 1989.

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