ECE 2646: Linear System Theory (3 Credits, Fall 2007)

Lecture 12: State Estimator

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State estimator

- Why do we need state estimator or state observer?

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State estimator

- Why do we need state estimator or state observer?
  - State feedback requires real-time values of state variables
  - State variables may not be accessible for direct connection in practice
  - Sensing devices or transducers may be unavailable or very expensive

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State estimator

- Why do we need state estimator or state observer?
- How to estimate state variables?

> Still remember this?
> If the state equation is observable, then
> $$\mathbf{x}(0) = \mathbf{W}_o^{-1}(t_1)\int_0^{t_1} e^{\mathbf{A}'t}\mathbf{C}'\overline{\mathbf{y}}(t)dt$$
> where
> $$\mathbf{W}_o(t) = \int_0^{t} e^{\mathbf{A}'\tau}\mathbf{C}'\mathbf{C}e^{\mathbf{A}\tau}d\tau,$$
> $$\overline{\mathbf{y}}(t) \equiv \mathbf{y}(t) - \mathbf{C}\int_0^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau - \mathbf{D}\mathbf{u}(t).$$

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State estimator

- Why do we need state estimator or state observer?
- How to estimate state variables?
  - State vector $\mathbf{x}(t)$ may be estimated from $\mathbf{u}$ and $\mathbf{y}$ over any time interval $[t, t{+}t_1]$

> Remark: If the state equation is observable, then
> $$\mathbf{x}(t) = \mathbf{W}_o^{-1}(t+t_1)\int_t^{t+t_1} e^{\mathbf{A}'\tau}\mathbf{C}'\overline{\mathbf{y}}(\tau)d\tau$$
> where
> $$\mathbf{W}_o(t+t_1) = \int_t^{t+t_1} e^{\mathbf{A}'\tau}\mathbf{C}'\mathbf{C}e^{\mathbf{A}\tau}d\tau,$$
> $$\overline{\mathbf{y}}(\tau) \equiv \mathbf{y}(\tau) - \mathbf{C}\int_t^{\tau} e^{\mathbf{A}(\tau-v)}\mathbf{B}\mathbf{u}(v)dv - \mathbf{D}\mathbf{u}(\tau).$$

> Question: What are the disadvantages of this method?

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State estimator

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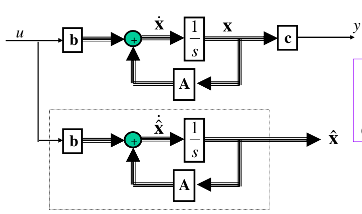
- How to estimate state variables?
  - State vector $\mathbf{x}(t)$ may be estimated from $\mathbf{u}$ and $\mathbf{y}$ over any time interval $[t, t{+}t_1]$
  - Open-loop estimator



> Question:
> What are the disadvantages of this method?

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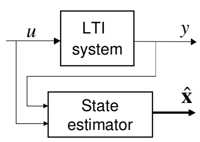
  - Open-loop estimator
  - Closed-loop estimator



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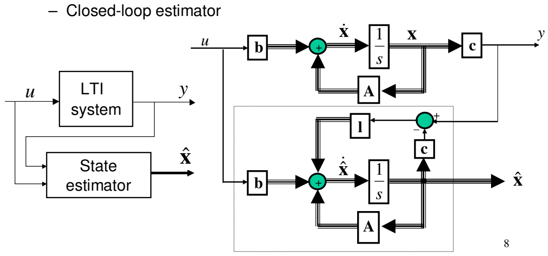

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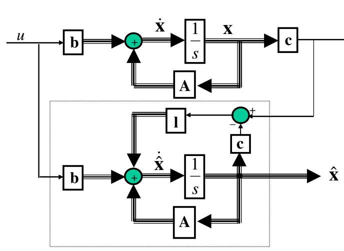

Question:
What are the equations for $\hat{\mathbf{x}}$ and $e \equiv \mathbf{x} - \hat{\mathbf{x}}$?

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Question: What are the equations for $\hat{\mathbf{x}}$ and $e \equiv \mathbf{x} - \hat{\mathbf{x}}$?

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{lc})\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}y$$
$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{lc})\mathbf{e}$$

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$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{lc})\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}y$$
$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{lc})\mathbf{e}$$

Theorem: All eigenvalues of $\mathbf{A} - \mathbf{lc}$ can be assigned arbitrarily (provided complex conjugate eigenvalues are assigned in pairs) by selecting a real constant vector $\mathbf{l}$ if and only if $(\mathbf{A}, \ \mathbf{c})$ is observable.

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$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{lc})\mathbf{e}$$

Question: Given a set of desired eigenvalues of $\mathbf{A} - \mathbf{lc}$, how to determine the vector $\mathbf{l}$?

Hint: Using duality.

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Example: Consider a system with the following state equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

$$y = [1 \quad 0]\mathbf{x}.$$

Design a state estimator with eigenvalues -10 and -10.

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$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{lc})\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}y$$

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{lc})\mathbf{e}$$

Remark: The gain $\mathbf{l}$ is normally chosen to make the dynamics of the estimator faster than those of the system; a rule of thumb often stated is to make the estimator two to four times faster than the system.

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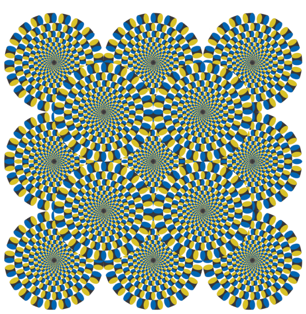

http://www.michaelbach.de/ot/mot_bounce/
http://www.michaelbach.de/ot/mot_feet_lin/index.html

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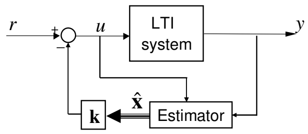
State estimator

- Why do we need state estimator or state observer?
- How to estimate state variables?
- Feedback from estimated states



$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad y = \mathbf{c}\mathbf{x}$$
$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}y$$
$$u = r - \mathbf{k}\hat{\mathbf{x}}$$

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$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad y = \mathbf{c}\mathbf{x}$$
$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}y$$
$$u = r - \mathbf{k}\hat{\mathbf{x}}$$

Questions:
(1) The eigenvalues of $\mathbf{A} - \mathbf{bk}$ are obtained from $u = r - \mathbf{kx}$. Do we still have the same set of eigenvalues in using estimated state variables?
(2) Will the eigenvalues of the estimator be affected by the connection?
(3) What is the effect of the estimator on the transfer function from r to y?

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State estimator

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$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad y = \mathbf{c}\mathbf{x}$$
$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{l}\mathbf{c})\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}y$$
$$u = r - \mathbf{k}\hat{\mathbf{x}}$$

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{bk} \\ \mathbf{lc} & \mathbf{A} - \mathbf{lc} - \mathbf{bk} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \end{bmatrix} r$$
$$y = [\mathbf{c} \quad \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{bk} & \mathbf{bk} \\ \mathbf{0} & \mathbf{A} - \mathbf{lc} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} r$$
$$y = [\mathbf{c} \quad \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

Question: What are the eigenvalues of these matrices?

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$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{bk} & \mathbf{bk} \\ \mathbf{0} & \mathbf{A}-\mathbf{lc} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} r$$

$$y = [\mathbf{c} \quad \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

Remarks:
(1) The eigenvalues are the union of those of $\mathbf{A} - \mathbf{bk}$ and $\mathbf{A} - \mathbf{lc}$.
(2) Inserting the state estimator does not affect the eigenvalues of the original state feedback; nor are the eigenvalues of the state estimator affected by the connection.
(3) The design of state feedback and the design of state estimator can be carried out independently. This is called the separation property.

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$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{bk} & \mathbf{bk} \\ \mathbf{0} & \mathbf{A}-\mathbf{lc} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} r$$

$$y = [\mathbf{c} \quad \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

Questions:
(1) Is the above system controllable?
(2) What is its transfer function?

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$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{bk} & \mathbf{bk} \\ \mathbf{0} & \mathbf{A}-\mathbf{lc} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} r$$

$$y = [\mathbf{c} \quad \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

Remark: The transfer function of the above system equals the transfer function of the original state feedback system without using a state estimator.

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References

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