ECE 2646: Linear System Theory (3 Credits, Fall 2007)

Lecture 11: State Feedback

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Outline of this lecture

- State feedback
- Regulation and tracking

State feedback

- Definition
  - Constant gain negative state feedback or, simply, state feedback

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u,\ \ \mathbf{y} = \mathbf{c}\mathbf{x}$$

$$\downarrow \quad u = r - \mathbf{k}\mathbf{x}$$

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{x} + \mathbf{b}r,\ \ \mathbf{y} = \mathbf{c}\mathbf{x}$$

State feedback

- Definition
- Controllability property is preserved in state feedback

Theorem: The pair $(\mathbf{A}-\mathbf{bk},\ \mathbf{b})$, for any 1 by n real constant vector $\mathbf{k}$, is controllable if and only if $(\mathbf{A},\ \mathbf{b})$ is controllable.

Question: How to prove this?

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State feedback

- Definition
- Controllability property is preserved in state feedback

Question: Is observability property preserved in state feedback?

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State feedback

- Definition
- Controllability property is preserved in state feedback

Theorem: Consider the state equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u,\ \mathbf{y} = \mathbf{c}\mathbf{x}$ with $n = 4$ and the characteristic polynomial

$$\Delta(s) = \det(s\mathbf{I} - \mathbf{A}) = s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4.$$

If the given state equation is controllable, then it can be transformed by the transformation $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ with

$$\mathbf{Q} \equiv \mathbf{P}^{-1} = [\mathbf{b}\ \ \mathbf{A}\mathbf{b}\ \ \mathbf{A}^2\mathbf{b}\ \ \mathbf{A}^3\mathbf{b}]\begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

into the controllable canonical form

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}u = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}u, \quad \mathbf{y} = \bar{\mathbf{c}}\bar{\mathbf{x}} = [\beta_1\ \ \beta_2\ \ \beta_3\ \ \beta_4]\bar{\mathbf{x}}.$$

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State feedback

- Definition
- Controllability property is preserved in state feedback

Theorem: Consider the state equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u,\ \mathbf{y} = \mathbf{c}\mathbf{x}$ with $n = 4$ and the characteristic polynomial

$$\Delta(s) = \det(s\mathbf{I} - \mathbf{A}) = s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4.$$

If the given state equation is controllable, then it can be transformed by the transformation $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ with

$$\mathbf{Q} \equiv \mathbf{P}^{-1} = [\mathbf{b}\ \ \mathbf{A}\mathbf{b}\ \ \mathbf{A}^2\mathbf{b}\ \ \mathbf{A}^3\mathbf{b}]\begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

into the controllable canonical form

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}u = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\bar{\mathbf{x}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}u, \quad \mathbf{y} = \bar{\mathbf{c}}\bar{\mathbf{x}} = [\beta_1\ \ \beta_2\ \ \beta_3\ \ \beta_4]\bar{\mathbf{x}}.$$

Remark: The theorem holds for every positive integer n.

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State feedback

- Definition
- Controllability property is preserved in state feedback
- Feedback transfer function and pole placement

Theorem: If $(\mathbf{A},\ \mathbf{b})$ is controllable, then by state feedback $u = r - \mathbf{k}\mathbf{x}$, where $\mathbf{k}$ is a 1 by n real constant vector, the eigenvalues of $\mathbf{A} - \mathbf{b}\mathbf{k}$ can arbitrarily be assigned provided that complex conjugate eigenvalues are assigned in pairs.

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State feedback

- Definition
- Controllability property is preserved in state feedback
- Feedback transfer function and pole placement

Question: How to find the state feedback gain $\mathbf{k}$ to obtain a set of desired eigenvalues?

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State feedback

- Definition
- Controllability property is preserved in state feedback
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Question: How to find the state feedback gain $\mathbf{k}$ to obtain a set of desired eigenvalues?

Remark: For the case of $n = 4$, from any set of desired eigenvalues, we can readily form

$$\Delta_f(s) = s^4 + \bar{\alpha}_1 s^3 + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s + \bar{\alpha}_4.$$

Let

$$\bar{\mathbf{k}} = [\bar{\alpha}_1 - \alpha_1 \;\; \bar{\alpha}_2 - \alpha_2 \;\; \bar{\alpha}_3 - \alpha_3 \;\; \bar{\alpha}_4 - \alpha_4], \quad \mathbf{k} = \bar{\mathbf{k}}\mathbf{P}$$

where

$$\mathbf{P}^{-1} = [\mathbf{b} \;\; \mathbf{Ab} \;\; \mathbf{A}^2\mathbf{b} \;\; \mathbf{A}^3\mathbf{b}]\begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

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State feedback

- Definition
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Example: Homework problem 8.1.

Exercise: Example 8.3 in the textbook.

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State feedback

- Definition
- Controllability property is preserved in state feedback
- Feedback transfer function and pole placement

Remark: State feedback can shift the poles of a plant but has no effect on the zeros. This can be used to explain why a state feedback may alter the observability property of a state equation. (How to explain?)

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State feedback

- Definition
- Controllability property is preserved in state feedback
- Feedback transfer function and pole placement

Remark: State feedback can shift the poles of a plant but has no effect on the zeros. This can be used to explain why a state feedback may alter the observability property of a state equation.

Question: In classical control, can we find a controller that may shift the zeros of a plant?

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State feedback

- Definition
- Controllability property is preserved in state feedback
- Feedback transfer function and pole placement
- Control system design using state feedback
  - Factors that we need to consider in the selection of a set of desired eigenvalues
    - Performance criteria, e.g., rise time, settling time, and overshoot
    - Zeros of the plant
    - Magnitude of actuating signals

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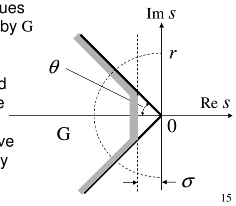
State feedback

- Definition
- Controllability property is preserved in state feedback
- Feedback transfer function and pole placement
- Control system design using state feedback
  - Factors that we need to consider in the selection of a set of desired eigenvalues
  - A rough guide for system design
    - We may place all eigenvalues inside the region denoted by G in the right figure
    - It is better to place all eigenvalues evenly around a circle with radius r inside the sector as shown
    - A final selection may involve compromises among many conflicting requirements



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Regulation and tracking

- Regulator problem
  - Suppose the reference signal r is zero, and the response of the system is caused by some nonzero initial conditions. The problem is to find a state feedback gain so that the response will die out at a desired rate

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Regulation and tracking

- Regulator problem
- Tracking problem
  - Suppose the reference signal r is a constant $r(t) = a$. The problem is to design an overall system so that $y(t)$ approaches $r(t)$ as t approaches infinity

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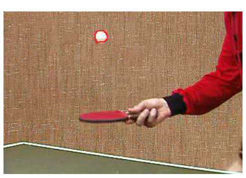
Regulation and tracking

- Regulator problem
- Tracking problem
- Servomechanism problem
  - The problem is to track a nonconstant reference signal



Tennis_Dog Tennis_Japanese Tennis_Fly

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