ECE 2646: Linear System Theory (3 Credits, Fall 2009)

Lecture 9: Coprimeness and Minimal Realizations

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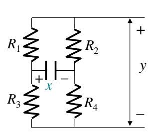
Instructor: Zhi-Hong Mao
Assistant Professor of ECE and Bioengineering
University of Pittsburgh, Pittsburgh, PA

Outline of this lecture

- Review of last lecture
- Coprimeness
- Minimal realizations

Review of last lecture

- Questions from last lecture

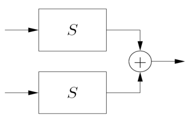


Question: Is the above system observable (the state variable is x)?

A state equation is said to be observable if for any unknown initial state $\mathbf{x}(0)$, there exists a finite $t_1 > 0$ such that the knowledge of the input $\mathbf{u}$ and the output $\mathbf{y}$ over $[0, t_1]$ suffices to determine uniquely the initial state $\mathbf{x}(0)$. Otherwise, equation is said to be unobservable.

Review of last lecture

- Questions from last lecture



$$S_1: \quad \dot{\mathbf{x}}_1 = \mathbf{A}_0\mathbf{x}_1 + \mathbf{B}_0\mathbf{u}_1$$
$$\mathbf{y}_1 = \mathbf{C}_0\mathbf{x}_1 + \mathbf{D}_0\mathbf{u}_1$$
$$S_2: \quad \dot{\mathbf{x}}_2 = \mathbf{A}_0\mathbf{x}_2 + \mathbf{B}_0\mathbf{u}_2$$
$$\mathbf{y}_2 = \mathbf{C}_0\mathbf{x}_2 + \mathbf{D}_0\mathbf{u}_2$$
$$\text{Output}: \ \mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$$

Question: What is the state-space equation for the above system (using $\mathbf{x} = [\mathbf{x}_1', \mathbf{x}_2']'$ and $\mathbf{u} = [\mathbf{u}_1', \mathbf{u}_2']'$)? Is the above system observable?

4

Review of last lecture

- Questions from last lecture
- Duality between controllability and observability

Theorem of Duality: The pair $(\mathbf{A}, \mathbf{B})$ is controllable if and only if the pair $(\mathbf{A}', \mathbf{B}')$ is observable.

5

Review of last lecture

- Questions from last lecture
- Duality between controllability and observability
- Canonical decomposition

Theorem: Consider the n-dimensional state equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$, with the controllability matrix C (i.e. $[\mathbf{B}\ \mathbf{AB} \ldots \mathbf{A}^{n-1}\mathbf{B}]$) of rank n_1, which is strictly less than n. We form the n by n matrix

$$\mathbf{P}^{-1} \equiv [\mathbf{q}_1 \cdots \mathbf{q}_{n_1} \cdots \mathbf{q}_n]$$

where the first n_1 columns are any n_1 linearly independent columns of C, and the remaining columns can arbitrarily be chosen as long as $\mathbf{P}$ is nonsingular. Then the equivalence transformation $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ will transform the original state equation into

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} \mathbf{u}, \ \mathbf{y} = [\bar{\mathbf{C}}_c \ \ \bar{\mathbf{C}}_{\bar{c}}] \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \mathbf{D}\mathbf{u},$$

where $\bar{\mathbf{A}}_c$ is n_1 by n_1 and $\bar{\mathbf{A}}_{\bar{c}}$ is $(n - n_1)$ by $(n - n_1)$, and the n_1-dimensional subequation of the derived equation,

$$\dot{\bar{\mathbf{x}}}_c = \bar{\mathbf{A}}_c\bar{\mathbf{x}}_c + \bar{\mathbf{B}}_c\mathbf{u}, \ \bar{\mathbf{y}} = \bar{\mathbf{C}}_c\bar{\mathbf{x}}_c + \mathbf{D}\mathbf{u},$$

is controllable and has the same transfer matrix as the original equation.

Why?

6

Review of last lecture

- Questions from last lecture
- Duality between controllability and observability
- Canonical decomposition

Question: Are the following two equations zero-state equivalent?

(1) $\begin{bmatrix}\dot{\bar{\mathbf{x}}}_c\\ \dot{\bar{\mathbf{x}}}_{\bar{c}}\end{bmatrix} = \begin{bmatrix}\bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12}\\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}}\end{bmatrix}\begin{bmatrix}\bar{\mathbf{x}}_c\\ \bar{\mathbf{x}}_{\bar{c}}\end{bmatrix} + \begin{bmatrix}\bar{\mathbf{B}}_c\\ \mathbf{0}\end{bmatrix}\mathbf{u},\ \mathbf{y} = [\bar{\mathbf{C}}_c \ \ \bar{\mathbf{C}}_{\bar{c}}]\begin{bmatrix}\bar{\mathbf{x}}_c\\ \bar{\mathbf{x}}_{\bar{c}}\end{bmatrix} + \mathbf{Du},$

(2) $\dot{\bar{\mathbf{x}}}_c = \bar{\mathbf{A}}_c\bar{\mathbf{x}}_c + \bar{\mathbf{B}}_c\mathbf{u},\ \bar{\mathbf{y}} = \bar{\mathbf{C}}_c\bar{\mathbf{x}}_c + \mathbf{Du}.$

7

Review of last lecture

- Questions from last lecture
- Duality between controllability and observability
- Canonical decomposition

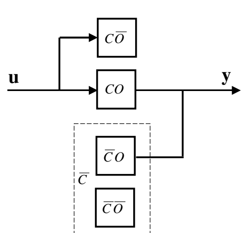
Example: Reduce the following state equation to a controllable one

$$\dot{\mathbf{x}} = \begin{bmatrix}0 & -0.5 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & -0.5 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}\mathbf{x} + \begin{bmatrix}0.5\\0\\0\\0\end{bmatrix}u,\quad y = [1\ \ 0\ \ 0\ \ 1]\mathbf{x} + u.$$

8

Review of last lecture

- Questions from last lecture
- Duality between controllability and observability
- Canonical decomposition
  - Kalman decomposition
    - Procedure: The state equation is first decomposed into controllable and uncontrollable subequations; then each subequation is decomposed into observable and unobservable parts
    - Only the controllable and observable part is connected to both the input and output terminals—thus the transfer matrix describes only this part of the system



9

Review of last lecture

- Questions from last lecture
- Duality between controllability and observability
- Canonical decomposition
  - Kalman decomposition

Theorem: Every state-space equation can be transformed, by an equivalence transformation, into the following canonical form

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_{co} \\ \dot{\bar{\mathbf{x}}}_{c\bar{o}} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}o} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_{co} & \mathbf{0} & \bar{\mathbf{A}}_{13} & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{c\bar{o}} & \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{43} & \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_{co} \\ \bar{\mathbf{B}}_{c\bar{o}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{u}, \quad \mathbf{y} = [\bar{\mathbf{C}}_{co} \ \mathbf{0} \ \bar{\mathbf{C}}_{\bar{c}o} \ \mathbf{0}]\bar{\mathbf{x}} + \mathbf{D}\mathbf{u},$$

where the vector $\bar{\mathbf{x}}_{co}$ is controllable and observable, $\bar{\mathbf{x}}_{c\bar{o}}$ is controllable but not observable, $\bar{\mathbf{x}}_{\bar{c}o}$ is observable but not controllable, and $\bar{\mathbf{x}}_{\bar{c}\bar{o}}$ is neither controllable nor observable. Furthermore, the original state equation is zero-state equivalent to the controllable and observable state equation

$$\dot{\bar{\mathbf{x}}}_{co} = \bar{\mathbf{A}}_{co}\bar{\mathbf{x}}_{co} + \bar{\mathbf{B}}_{co}\mathbf{u}, \quad \bar{\mathbf{y}} = \bar{\mathbf{C}}_{co}\bar{\mathbf{x}}_{co} + \mathbf{D}\mathbf{u},$$

and has the transfer matrix

$$\hat{\mathbf{G}}(s) = \bar{\mathbf{C}}_{co}(s\mathbf{I} - \bar{\mathbf{A}}_{co})^{-1}\bar{\mathbf{B}}_{co} + \mathbf{D}.$$

Double and triple inverted pendulum

Coprimeness

- Controllable canonical form

Exercise: Find a realization in controllable canonical form for the following transfer function:

$$\hat{g}(s) = \frac{N(s)}{D(s)} = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4}.$$

Coprimeness

- Controllable canonical form

Exercise: Check the controllability of the following state-space equation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}u,$$

$$y = \mathbf{c}\mathbf{x} = [\beta_1\ \beta_2\ \beta_3\ \beta_4]\mathbf{x}.$$

Coprimeness

- Controllable canonical form

Another controllable canonical form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_4 & -\alpha_3 & -\alpha_2 & -\alpha_1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}u,$$

$$y = [\beta_4\ \beta_3\ \beta_2\ \beta_1]\mathbf{x}.$$

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}u,$$

$$y = \mathbf{c}\mathbf{x} = [\beta_1\ \beta_2\ \beta_3\ \beta_4]\mathbf{x}.$$

Coprimeness

- Controllable canonical form

Exercise: Check the observability of the following state-space equation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u = \begin{bmatrix} -\alpha_1 & -\alpha_2 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u,$$

$$y = \mathbf{c}\mathbf{x} = [\beta_1\ \beta_2]\mathbf{x},$$

which is a realization (in controllable canonical form) of the following transfer function:

$$\hat{g}(s) = \frac{N(s)}{D(s)} = \frac{\beta_1 s + \beta_2}{s^2 + \alpha_1 s + \alpha_2}.$$

Coprimeness

- Controllable canonical form

> Exercise: Check the observability of the following state-space equation:
>
> $$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u = \begin{bmatrix} -\alpha_1 & -\alpha_2 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u,$$
>
> $$y = \mathbf{c}\mathbf{x} = [\beta_1 \ \beta_2]\mathbf{x},$$
>
> which is a realization (in controllable canonical form) of the following transfer function:
>
> $$\hat{g}(s) = \frac{N(s)}{D(s)} = \frac{\beta_1 s + \beta_2}{s^2 + \alpha_1 s + \alpha_2}.$$

> Observations: The observability depends on whether $N(s)$ and $D(s)$ have common root(s), in other words, whether they are coprime.

16

Coprimeness

- Controllable canonical form
- Coprimeness
  - Definitions
    - Two polynomials are said to be coprime if they have no common factor of degree at least 1
    - A polynomial is called a greatest common divisor (gcd) of $D(s)$ and $N(s)$ if (1) it is a common factor of $D(s)$ and $N(s)$ and (2) it can be divided without remainder by every other common divisor of $D(s)$ and $N(s)$
    - Two polynomials are coprime if their gcd is a nonzero constant, a polynomial of degree 0; they are not coprime if their gcd has degree 1 or higher

17

Coprimeness

- Controllable canonical form
- Coprimeness

> **Theorem:** The controllable canonical form for $N(s)/D(s)$ is observable if and only if $D(s)$ and $N(s)$ are coprime.

18

Coprimeness

- Controllable canonical form
- Coprimeness
- **Observable canonical form**

$$g(s)=\frac{N(s)}{D(s)}=\frac{\beta_1 s^3+\beta_2 s^2+\beta_3 s+\beta_4}{s^4+\alpha_1 s^3+\alpha_2 s^2+\alpha_3 s+\alpha_4}$$

$$\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}+\mathbf{b}u=\begin{bmatrix}-\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\mathbf{x}+\begin{bmatrix}1\\0\\0\\0\end{bmatrix}u,$$

$$y=\mathbf{c}\mathbf{x}=[\beta_1\ \beta_2\ \beta_3\ \beta_4]\mathbf{x}$$

$$\hat{g}(s)=\hat{g}'(s)=[\mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}]'=\mathbf{b}'(s\mathbf{I}-\mathbf{A}')^{-1}\mathbf{c}'$$

$$\dot{\mathbf{x}}=\mathbf{A}'\mathbf{x}+\mathbf{c}'u=\begin{bmatrix}-\alpha_1 & 1 & 0 & 0\\ -\alpha_2 & 0 & 1 & 0\\ -\alpha_3 & 0 & 0 & 1\\ -\alpha_4 & 0 & 0 & 0\end{bmatrix}\mathbf{x}+\begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\end{bmatrix}u,$$

$$y=\mathbf{b}'\mathbf{x}=[1\ 0\ 0\ 0]\mathbf{x}$$

← Observable canonical form

19

Coprimeness

- Controllable canonical form
- Coprimeness
- **Observable canonical form**

Another observable canonical form:

$$\dot{\mathbf{x}}=\begin{bmatrix}0 & 0 & 0 & -\alpha_4\\ 1 & 0 & 0 & -\alpha_3\\ 0 & 1 & 0 & -\alpha_2\\ 0 & 0 & 1 & -\alpha_1\end{bmatrix}\mathbf{x}+\begin{bmatrix}\beta_4\\ \beta_3\\ \beta_2\\ \beta_1\end{bmatrix}u,$$

$$y=[0\ 0\ 0\ 1]\mathbf{x}$$

$$\mathbf{P}=\begin{bmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}$$

$$\dot{\mathbf{x}}=\begin{bmatrix}-\alpha_1 & 1 & 0 & 0\\ -\alpha_2 & 0 & 1 & 0\\ -\alpha_3 & 0 & 0 & 1\\ -\alpha_4 & 0 & 0 & 0\end{bmatrix}\mathbf{x}+\begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\end{bmatrix}u,$$

$$y=[1\ 0\ 0\ 0]\mathbf{x}$$

20

Coprimeness

- Controllable canonical form
- Coprimeness
- **Observable canonical form**

Theorem: The observable canonical form for $N(s)/D(s)$ is controllable if and only if $D(s)$ and $N(s)$ are coprime.

21

Minimal realizations

- What are minimal realizations and why use them?
  - Definition
    - Realizations with smallest possible dimension are called minimal-dimensional or minimal realizations
  - Why use realizations and minimal realizations?
    - Many design methods and computational algorithms are developed for state equations
    - Once a transfer function is realized into a state equation, the transfer function can be implemented using op-amp circuits
    - If we use a minimal realization to implement a transfer function, then the scale of the implementation will be minimum (e.g. the number of integrators used in an op-amp circuits will be minimum)

22

Minimal realizations

- What are minimal realizations and why use them?
- Fraction, coprime fraction, and degree of a proper rational function
  - Let $N(s)$ and $D(s)$ be two polynomials. Then we call $N(s)/D(s)$ a polynomial fraction, or simply, a fraction
  - Consider a proper rational function $\hat{g}(s) = N(s)/D(s)$. Let $R(s)$ be a greatest common divisor (gcd) of $N(s)$ and $D(s)$. That is, we can write $N(s) = \bar{N}(s)R(s)$ and $D(s) = \bar{D}(s)R(s)$. Then $\hat{g}(s)$ can be reduced to $\bar{N}(s)/\bar{D}(s)$, which is called a coprime fraction. We call $\bar{D}(s)$ a characteristic polynomial of $\hat{g}(s)$. The degree of this characteristic polynomial is defined as the degree of $\hat{g}(s)$

Example: $$\hat{g}(s) = \frac{s^2-1}{4(s^3-1)}.$$

23

Minimal realizations

- What are minimal realizations and why use them?
- Fraction, coprime fraction, and degree of a proper rational function
- Properties of minimal realizations of a transfer function

Theorem: A state equation $(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ is a minimal realization of a rational function $\hat{g}(s)$ if and only if $(\mathbf{A}, \mathbf{b})$ is controllable and $(\mathbf{A}, \mathbf{c})$ is observable or if and only if

$$\dim \mathbf{A} = \deg \hat{g}(s).$$

24

Minimal realizations

- What are minimal realizations and why use them?
- Fraction, coprime fraction, and degree of a proper rational function
- Properties of minimal realizations of a transfer function

Theorem: A state equation $(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ is a minimal realization of a rational function $\hat{g}(s)$ if and only if $(\mathbf{A}, \mathbf{b})$ is controllable and $(\mathbf{A}, \mathbf{c})$ is observable or if and only if

$$\dim \mathbf{A} = \deg \hat{g}(s).$$

Remarks:
(1) The theorem provides an alternative way of checking controllability and observability. (How to check?)
(2) If $\hat{g}(s) = N(s)/D(s)$, where $N(s)$ and $D(s)$ are coprime, and $(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ is a minimal realization of $\hat{g}(s)$, then

$$D(s) = k \det(s\mathbf{I} - \mathbf{A})$$

for some nonzero k.

25

Minimal realizations

- What are minimal realizations and why use them?
- Fraction, coprime fraction, and degree of a proper rational function
- Properties of minimal realizations of a transfer function

Theorem: A state equation $(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ is a minimal realization of a rational function $\hat{g}(s)$ if and only if $(\mathbf{A}, \mathbf{b})$ is controllable and $(\mathbf{A}, \mathbf{c})$ is observable or if and only if

$$\dim \mathbf{A} = \deg \hat{g}(s).$$

Remarks:
(3) If $(\mathbf{A}, \mathbf{b}, \mathbf{c}, d)$ is controllable and observable, then we have "asymptotic stability ⇔ BIBO stability." (Why?)
(4) Controllable and observable state equations and coprime fractions contain essentially the same information and either description can be used to carry out analysis and design.

26

Minimal realizations

- What are minimal realizations and why use them?
- Fraction, coprime fraction, and degree of a proper rational function
- Properties of minimal realizations of a transfer function

Theorem: All minimal realizations of $\hat{g}(s)$ are equivalent.

Remark: The theorem provides an alternative way of checking equivalence. (How to check?)

27

References

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