ECE 2646: Linear System Theory (3 Credits, Fall 2009)

Lecture 8: Observability and Canonical Decomposition

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Outline of this lecture

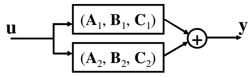
- Homework 4
- An exercise
- An example of controllability analysis
- Observability
- Canonical decomposition

Homework 4

- Problems 6.1 and 6.2 (for these two problems, only solve the observability parts), 6.8, 7.1, 7.2, and 7.12
- Due Wednesday 11/11 (two weeks later)

An exercise

Consider the parallel connection of two systems ($\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{C}_1$) and ($\mathbf{A}_2$, $\mathbf{B}_2$, $\mathbf{C}_2$), with states $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively, as shown below:



(a) Write state equations for the parallel connection system using the state $\mathbf{x} = [\mathbf{x}_1{}' \ \mathbf{x}_2{}']'$.
(b) Show that the parallel connection is uncontrollable if either subsystem is uncontrollable.
(c) If both subsystems are controllable, is their parallel connection controllable?

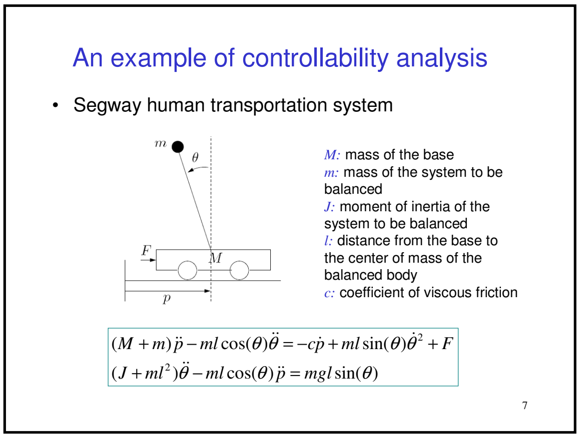
An example of controllability analysis

- Segway human transportation system

A question concerning controllability: Can we move from one stationary point to another by appropriate application of forces through the wheels?

An example of controllability analysis

- Segway human transportation system

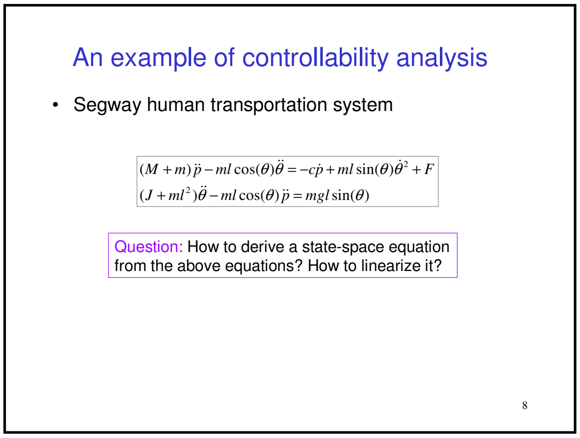


An example of controllability analysis

- Segway human transportation system

M: mass of the base
m: mass of the system to be balanced
J: moment of inertia of the system to be balanced
l: distance from the base to the center of mass of the balanced body
c: coefficient of viscous friction

$$(M+m)\ddot{p} - ml\cos(\theta)\ddot{\theta} = -c\dot{p} + ml\sin(\theta)\dot{\theta}^2 + F$$
$$(J+ml^2)\ddot{\theta} - ml\cos(\theta)\ddot{p} = mgl\sin(\theta)$$

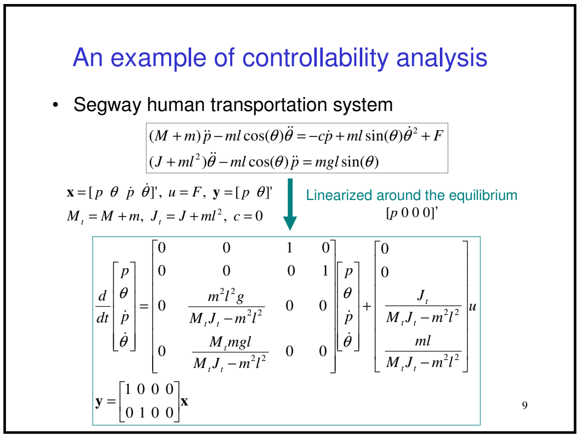


An example of controllability analysis

- Segway human transportation system

$$(M+m)\ddot{p} - ml\cos(\theta)\ddot{\theta} = -c\dot{p} + ml\sin(\theta)\dot{\theta}^2 + F$$
$$(J+ml^2)\ddot{\theta} - ml\cos(\theta)\ddot{p} = mgl\sin(\theta)$$

Question: How to derive a state-space equation from the above equations? How to linearize it?



An example of controllability analysis

- Segway human transportation system

$$(M+m)\ddot{p} - ml\cos(\theta)\ddot{\theta} = -c\dot{p} + ml\sin(\theta)\dot{\theta}^2 + F$$
$$(J+ml^2)\ddot{\theta} - ml\cos(\theta)\ddot{p} = mgl\sin(\theta)$$

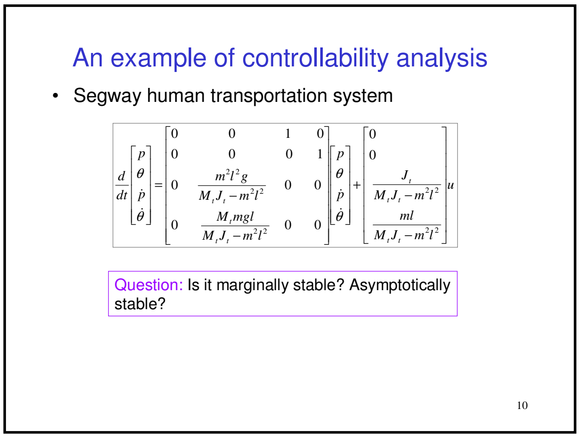
$\mathbf{x} = [p\ \ \theta\ \ \dot{p}\ \ \dot{\theta}]',\ u = F,\ \mathbf{y} = [p\ \ \theta]'$

$M_t = M + m,\ J_t = J + ml^2,\ c = 0$

$\big\downarrow$ Linearized around the equilibrium $[p\ 0\ 0\ 0]'$

$$\frac{d}{dt}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{m^2l^2g}{M_tJ_t - m^2l^2} & 0 & 0 \\ 0 & \dfrac{M_tmgl}{M_tJ_t - m^2l^2} & 0 & 0 \end{bmatrix}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{J_t}{M_tJ_t - m^2l^2} \\ \dfrac{ml}{M_tJ_t - m^2l^2} \end{bmatrix}u$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

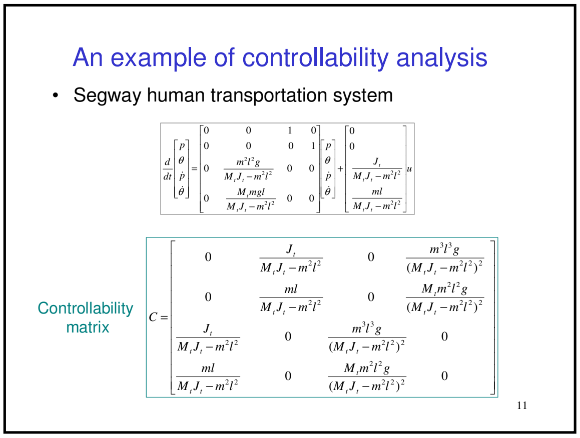


An example of controllability analysis

- Segway human transportation system

$$\frac{d}{dt}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2l^2g}{M_tJ_t - m^2l^2} & 0 & 0 \\ 0 & \frac{M_tmgl}{M_tJ_t - m^2l^2} & 0 & 0 \end{bmatrix}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{M_tJ_t - m^2l^2} \\ \frac{ml}{M_tJ_t - m^2l^2} \end{bmatrix}u$$

Question: Is it marginally stable? Asymptotically stable?



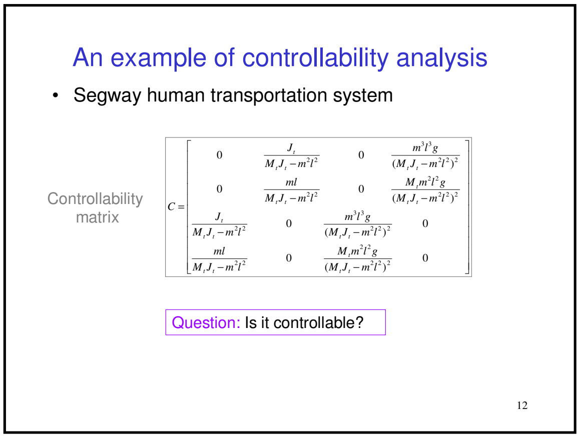
An example of controllability analysis

- Segway human transportation system

$$\frac{d}{dt}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2l^2g}{M_tJ_t - m^2l^2} & 0 & 0 \\ 0 & \frac{M_tmgl}{M_tJ_t - m^2l^2} & 0 & 0 \end{bmatrix}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{M_tJ_t - m^2l^2} \\ \frac{ml}{M_tJ_t - m^2l^2} \end{bmatrix}u$$

Controllability matrix

$$C = \begin{bmatrix} 0 & \frac{J_t}{M_tJ_t - m^2l^2} & 0 & \frac{m^3l^3g}{(M_tJ_t - m^2l^2)^2} \\ 0 & \frac{ml}{M_tJ_t - m^2l^2} & 0 & \frac{M_tm^2l^2g}{(M_tJ_t - m^2l^2)^2} \\ \frac{J_t}{M_tJ_t - m^2l^2} & 0 & \frac{m^3l^3g}{(M_tJ_t - m^2l^2)^2} & 0 \\ \frac{ml}{M_tJ_t - m^2l^2} & 0 & \frac{M_tm^2l^2g}{(M_tJ_t - m^2l^2)^2} & 0 \end{bmatrix}$$



An example of controllability analysis

- Segway human transportation system

Controllability matrix

$$C = \begin{bmatrix} 0 & \frac{J_t}{M_tJ_t - m^2l^2} & 0 & \frac{m^3l^3g}{(M_tJ_t - m^2l^2)^2} \\ 0 & \frac{ml}{M_tJ_t - m^2l^2} & 0 & \frac{M_tm^2l^2g}{(M_tJ_t - m^2l^2)^2} \\ \frac{J_t}{M_tJ_t - m^2l^2} & 0 & \frac{m^3l^3g}{(M_tJ_t - m^2l^2)^2} & 0 \\ \frac{ml}{M_tJ_t - m^2l^2} & 0 & \frac{M_tm^2l^2g}{(M_tJ_t - m^2l^2)^2} & 0 \end{bmatrix}$$

Question: Is it controllable?

An example of controllability analysis

- Segway human transportation system

Controllability matrix

$$C = \begin{bmatrix} 0 & \frac{J_t}{M_t J_t - m^2 l^2} & 0 & \frac{m^3 l^3 g}{(M_t J_t - m^2 l^2)^2} \\ 0 & \frac{ml}{M_t J_t - m^2 l^2} & 0 & \frac{M_t m^2 l^2 g}{(M_t J_t - m^2 l^2)^2} \\ \frac{J_t}{M_t J_t - m^2 l^2} & 0 & \frac{m^3 l^3 g}{(M_t J_t - m^2 l^2)^2} & 0 \\ \frac{ml}{M_t J_t - m^2 l^2} & 0 & \frac{M_t m^2 l^2 g}{(M_t J_t - m^2 l^2)^2} & 0 \end{bmatrix}$$

$$\boxed{\left|\det(C)\right| = \frac{m^4 l^4 g^2}{(M_t J_t - m^2 l^2)^4} \neq 0} \longrightarrow$$

The system is controllable!

An example of controllability analysis

- Segway human transportation system

Observability

- Definition
  - – The concept of observability is dual to that of controllability
  - – Observability studies the possibility of estimating the state from the output

  (Reminder: Controllability studies the possibility of steering the state from the input)

Observability

- Definition
  - The concept of observability is dual to that of controllability
  - Observability studies the possibility of estimating the state from the output
  - Consider an n-dimensional p-input q-output state equation
    $$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$
    $$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}.$$

    This state equation is said to be observable if for any unknown initial state $\mathbf{x}(0)$, there exists a finite $t_1 > 0$ such that the knowledge of the input $\mathbf{u}$ and the output $\mathbf{y}$ over $[0, t_1]$ suffices to determine uniquely the initial state $\mathbf{x}(0)$. Otherwise, equation is said to be unobservable

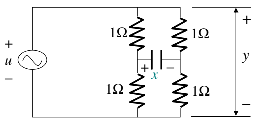
    (Reminder: The state equation is said to be controllable if for any initial state $\mathbf{x}(0) = \mathbf{x}_0$ and any final state $\mathbf{x}_1$, there exists an input that transfer $\mathbf{x}_0$ to $\mathbf{x}_1$ in a finite time)

Observability

- Definition
  - The concept of observability is dual to that of controllability
  - Observability studies the possibility of estimating the state from the output and input
  - Observable and unobservable
  - Significance of the concept of observability
    - If a system is observable, then there are no “hidden” dynamics inside it; we can understand everything that is going on through observation (over time) of the inputs and outputs
    - The problem of observability is of significant practical interest because it will tell if a set of sensors are sufficient for controlling a system

Observability

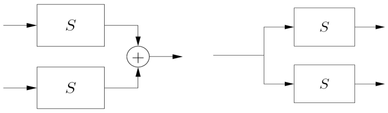
- Definition
- Examples of observable and unobservable systems



Question: Is the above system observable (the state variable is x)?

Observability

- Definition
- Examples of observable and unobservable systems



Question: Is the left system observable?
(Reminder: Is the right system controllable?)

Observability

- Definition
- Examples of observable and unobservable systems

Diagnostic techniques in traditional Chinese medicine

Detection of drowsiness

Observability

- Definition
- Examples of observable and unobservable systems
- Theorems

The state equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$
is observable if and only if the matrix

$$\mathbf{W}_o(t) = \int_0^t e^{\mathbf{A}'\tau}\mathbf{C}'\mathbf{C}e^{\mathbf{A}\tau}d\tau$$

is nonsingular for any $t > 0$.

Reminder: The pair $(\mathbf{A}, \mathbf{B})$ is controllable if and only if the matrix

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}'e^{\mathbf{A}'\tau}d\tau = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}'e^{\mathbf{A}'(t-\tau)}d\tau$$

is nonsingular for any $t > 0$.

Observability

- Definition
- Examples of observable and unobservable systems
- Theorems

$$\mathbf{W}_o(t) = \int_0^t e^{\mathbf{A}'\tau}\mathbf{C}'\mathbf{C}e^{\mathbf{A}\tau}d\tau$$

Remark: If the state equation is observable, then

$$\mathbf{x}(0) = \mathbf{W}_o^{-1}(t_1)\int_0^{t_1} e^{\mathbf{A}'t}\mathbf{C}'\bar{\mathbf{y}}(t)dt$$

where

$$\bar{\mathbf{y}}(t) \equiv \mathbf{y}(t) - \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau - \mathbf{D}\mathbf{u}(t).$$

Remark: Observability depends only on $\mathbf{A}$ and $\mathbf{C}$. Thus observability is a property of the pair $(\mathbf{A}, \mathbf{C})$ and is independent of $\mathbf{B}$ and $\mathbf{D}$.

Observability

- Definition
- Examples of observable and unobservable systems
- Theorems

Theorem of duality: The pair $(\mathbf{A}, \mathbf{B})$ is controllable if and only if the pair $(\mathbf{A}', \mathbf{B}')$ is observable.

Question: Can you prove the theorem of duality?

Hint: $\mathbf{W}_o(t) = \int_0^t e^{\mathbf{A}'\tau}\mathbf{C}'\mathbf{C}e^{\mathbf{A}\tau}d\tau,\quad \mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}'e^{\mathbf{A}'\tau}d\tau.$

Observability

- Definition
- Examples of observable and unobservable systems
- Theorems

The n-dimensional pair $(\mathbf{A}, \mathbf{C})$, where $\mathbf{A}$ and $\mathbf{C}$ are n by n and q by n matrices respectively, is observable if and only if the nq by n observability matrix

$$O = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}$$

has rank n (full column rank).

Question: Can you prove this?

Observability

- Definition
- Examples of observable and unobservable systems
- Theorems

The n-dimensional pair $(\mathbf{A}, \mathbf{C})$ is observable if and only if the matrix

$$O_{n-q+1} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-q} \end{bmatrix}$$

where the rank of $\mathbf{C}$ is q, has rank n or the n by n matrix $O'_{n-q+1}O_{n-q+1}$ is nonsingular.

Observability

- Definition
- Examples of observable and unobservable systems
- Theorems

The observability property is invariant under any equivalence transformation.

Canonical Decomposition

- Stability, controllability, and observability are preserved under equivalence transform

$$\begin{tikzcd}
\boxed{\begin{array}{l} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} = \mathbf{Cx} + \mathbf{Du} \end{array}} \arrow[r, "\bar{\mathbf{x}} = \mathbf{Px}"] & \boxed{\begin{array}{l} \dot{\bar{\mathbf{x}}} = \overline{\mathbf{A}}\bar{\mathbf{x}} + \overline{\mathbf{B}}\mathbf{u} \\ \mathbf{y} = \overline{\mathbf{C}}\bar{\mathbf{x}} + \overline{\mathbf{D}}\mathbf{u} \end{array}} \arrow[d] \\
& \boxed{\begin{array}{c} \overline{\mathbf{A}} = \mathbf{PAP}^{-1}, \overline{\mathbf{B}} = \mathbf{PB}, \overline{\mathbf{C}} = \mathbf{CP}^{-1}, \overline{\mathbf{D}} = \mathbf{D} \\ \overline{C} = \mathbf{P}C,\ \overline{O} = O\mathbf{P}^{-1} \end{array}}
\end{tikzcd}$$

Canonical Decomposition

- Stability, controllability, and observability are preserved under equivalence transform
- Controllable and uncontrollable subspaces

Consider the n-dimensional state equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$, with the controllability matrix C (i.e. $[\mathbf{B}\ \mathbf{A}\mathbf{B}\ \dots\ \mathbf{A}^{n\text{-}1}\mathbf{B}]$) of rank n_1, which is strictly less than n. We form the n by n matrix

$$\mathbf{P}^{-1} \equiv [\mathbf{q}_1 \cdots \mathbf{q}_{n_1} \cdots \mathbf{q}_n]$$

where the first n_1 columns are any n_1 linearly independent columns of C, and the remaining columns can arbitrarily be chosen as long as $\mathbf{P}$ is nonsingular. Then the equivalence transformation $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ will transform the original state equation into

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} \mathbf{u},\ \ \mathbf{y} = [\bar{\mathbf{C}}_c\ \ \bar{\mathbf{C}}_{\bar{c}}] \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \mathbf{D}\mathbf{u},$$

where $\bar{\mathbf{A}}_c$ is n_1 by n_1 and $\bar{\mathbf{A}}_{\bar{c}}$ is $(n - n_1)$ by $(n - n_1)$, and the n_1-dimensional subequation of the derived equation,

$$\dot{\bar{\mathbf{x}}}_c = \bar{\mathbf{A}}_c \bar{\mathbf{x}}_c + \bar{\mathbf{B}}_c \mathbf{u},\ \ \bar{\mathbf{y}} = \bar{\mathbf{C}}_c \bar{\mathbf{x}}_c + \mathbf{D}\mathbf{u},$$

is controllable and has the same transfer matrix as the original equation.

Canonical Decomposition

- Stability, controllability, and observability are preserved under equivalence transform
- Controllable and uncontrollable subspaces

Example: Reduce the following state equation to a controllable one

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u,\ \ y = [1\ 1\ 1]\mathbf{x}.$$

Canonical Decomposition

- Stability, controllability, and observability are preserved under equivalence transform
- Controllable and uncontrollable subspaces
- Observable and unobservable subspaces

Consider the n-dimensional state equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$, with the observability matrix O of rank n_2, which is strictly less than n. We form the n by n matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_{n_2} \\ \vdots \\ \mathbf{p}_n \end{bmatrix}$$

where the first n_2 rows are any n_2 linearly independent rows of O, and the remaining rows can arbitrarily be chosen as long as $\mathbf{P}$ is nonsingular.

Then the equivalence transformation $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ will transform the original state equation into

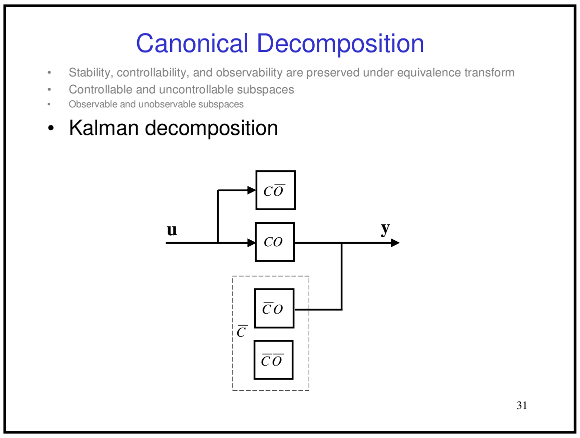
$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_o \\ \dot{\bar{\mathbf{x}}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_o & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_o \\ \bar{\mathbf{B}}_{\bar{o}} \end{bmatrix} \mathbf{u},$$

$$\mathbf{y} = [\bar{\mathbf{C}}_o\ \ \mathbf{0}] \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \mathbf{D}\mathbf{u},$$

where $\bar{\mathbf{A}}_o$ is n_2 by n_2 and $\bar{\mathbf{A}}_{\bar{o}}$ is $(n - n_2)$ by $(n - n_2)$, and the n_2-dimensional subequation of the derived equation,

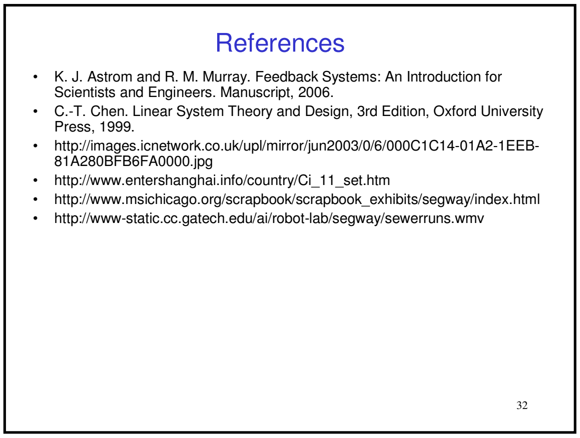
$$\dot{\bar{\mathbf{x}}}_o = \bar{\mathbf{A}}_o \bar{\mathbf{x}}_o + \bar{\mathbf{B}}_o \mathbf{u},\ \ \bar{\mathbf{y}} = \bar{\mathbf{C}}_o \bar{\mathbf{x}}_o + \mathbf{D}\mathbf{u},$$

is observable and has the same transfer matrix as the original equation.



Canonical Decomposition

- Stability, controllability, and observability are preserved under equivalence transform
- Controllable and uncontrollable subspaces
- Observable and unobservable subspaces
- Kalman decomposition



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