ECE 2646: Linear System Theory (3 Credits, Fall 2009)

Lecture 7: Controllability

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Outline of this lecture

- Announcement
- Review of last lecture
- Controllability

Announcement

- I will attend a conference next Tuesday and Wednesday
- Office hours this week and next week
  - Friday 10/16: 3 pm—6 pm
  - Monday 10/19: 4 pm—7 pm

Review of last lecture

- Realizations

Question:
(1) What does “realizable” mean?
(2) What is the condition for a transfer matrix to be realizable?

Review of last lecture

- Realizations
  - – Procedure to obtain controllable canonical form

Step 1:

$$\hat{\mathbf{G}}(s) = \hat{\mathbf{G}}(\infty) + \hat{\mathbf{G}}_{sp}(s),$$

where $\hat{\mathbf{G}}(s)$ is a $q \times p$ proper rational matrix
and $\hat{\mathbf{G}}_{sp}(s)$ is the strictly proper part of $\hat{\mathbf{G}}(s)$

Step 2:

$$\hat{\mathbf{G}}_{sp}(s) = \frac{1}{d(s)}[\mathbf{N}(s)] = \frac{1}{s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r}[\mathbf{N}_1 s^{r-1} + \mathbf{N}_2 s^{r-2} + \cdots + \mathbf{N}_{r-1}s + \mathbf{N}_r],$$

where $d(s)$ is the least common denominator of all entries of $\hat{\mathbf{G}}_{sp}(s)$,
it is monic, i.e., with 1 as its leading coefficient,
and $\mathbf{N}_i$ are $q \times p$ constant matrices

Step 3:

$$\dot{\mathbf{x}} = \begin{bmatrix} -\alpha_1\mathbf{I}_p & -\alpha_2\mathbf{I}_p & \cdots & -\alpha_{r-1}\mathbf{I}_p & -\alpha_r\mathbf{I}_p \\ \mathbf{I}_p & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_p & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_p & \mathbf{0} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \mathbf{I}_p \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}u$$

$$\mathbf{y} = [\mathbf{N}_1 \ \ \mathbf{N}_2 \ \ \cdots \ \ \mathbf{N}_{r-1} \ \ \mathbf{N}_r]\mathbf{x} + \hat{\mathbf{G}}(\infty)\mathbf{u}$$

Review of last lecture

- Realizations
- BIBO stability

Question: How to verify that a system is BIBO stable?

A system is said to be BIBO stable if every bounded input excites a bounded output.

An SISO system is BIBO stable if and only if its impulse response $g(t)$ is absolutely integrable in $[0, \infty)$, or

$$\int_0^\infty |g(t)|dt \le M < \infty$$

For some constant M.

An SISO system with proper rational transfer function is BIBO stable if and only if every pole of the transfer function has a negative real part or, equivalently, lies inside the left-half s-plane.

Review of last lecture

- Realizations
- BIBO stability

Question: If $H(s)$ is BIBO stable, is a system with transfer function $e^{-2s}H(s)$ BIBO stable?

Review of last lecture

- Realizations
- BIBO stability
- Internal stability

Question:

What are the definitions of “marginally stable” and “asymptotically stable”?

How to verify that a system is marginally stable or asymptotically stable?

Review of last lecture

- Realizations
- BIBO stability
- Internal stability

If $\mathbf{A}$ has distinct eigenvalues, then the equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is marginally stable if and only if all eigenvalues of $\mathbf{A}$ have zero or negative real parts.

The equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is asymptotically stable if and only all eigenvalues of $\mathbf{A}$ have negative real parts.

Review of last lecture

- Realizations
- BIBO stability
- Internal stability

Question:

Does “asymptotically stable” imply “BIBO stable”?

Does “BIBO stable” imply “asymptotically stable”?

Does “marginally stable” imply “BIBO stable”? (Hint: Consider the integrator.)

Does “BIBO stable” imply “marginally stable”?

Controllability

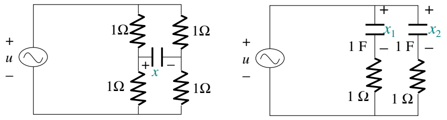
- Definition
  - Controllability deals with whether or not the state of a state-space equation can be controlled from the input
  - Consider a state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}.$$

This state equation or the pair $(\mathbf{A}, \mathbf{B})$ is said to be controllable if for any initial state $\mathbf{x}(0) = \mathbf{x}_0$ and any final state $\mathbf{x}_1$, there exists an input that transfer $\mathbf{x}_0$ to $\mathbf{x}_1$ in a finite time. Otherwise the state equation or $(\mathbf{A}, \mathbf{B})$ is said to be uncontrollable

Controllability

- Definition
- Examples of controllable and uncontrollable systems



Question: Are the above systems controllable (in the left system, the state variable is x; in the right system, the state variables are x_1 and x_2?

Controllability

- Definition
- Examples of controllable and uncontrollable systems

Heart rate control, yoga practitioner, and implanted heart

Controllability

- Definition
- Examples of controllable and uncontrollable systems

Yoked eye movements and independent eye movements (in chameleon and sandlance)

Controllability

- Definition
- Examples of controllable and uncontrollable systems
- Theorems

The pair $(\mathbf{A}, \mathbf{B})$ is controllable if and only if the matrix

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}' e^{\mathbf{A}'\tau} d\tau = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}' e^{\mathbf{A}'(t-\tau)} d\tau$$

is nonsingular for any $t > 0$.

Controllability

- Definition
- Examples of controllable and uncontrollable systems
- Theorems

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}' e^{\mathbf{A}'\tau} d\tau = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}' e^{\mathbf{A}'(t-\tau)} d\tau$$

Remark: If the pair $(\mathbf{A}, \mathbf{B})$ is controllable, then for any $\mathbf{x}(0) = \mathbf{x}_0$ and any $\mathbf{x}(t_1) = \mathbf{x}_1$, the input

$$\mathbf{u}(t) = -\mathbf{B}' e^{\mathbf{A}'(t_1 - t)} \mathbf{W}_c^{-1}(t_1)[e^{\mathbf{A}t_1}\mathbf{x}_0 - \mathbf{x}_1]$$

will transfer $\mathbf{x}_0$ to $\mathbf{x}_1$ at time t_1.

Question: Please verify that

$$\mathbf{x}(t_1) = e^{\mathbf{A}t_1}\mathbf{x}(0) + \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\mathbf{u}(\tau) d\tau = \mathbf{x}_1.$$

Controllability

- Definition
- Examples of controllable and uncontrollable systems
- Theorems

The n-dimensional pair $(\mathbf{A}, \mathbf{B})$, where $\mathbf{A}$ and $\mathbf{B}$ are n by n and n by p matrices respectively, is controllable if and only if the n by np controllability matrix

$$C = [\mathbf{B}\ \ \mathbf{AB}\ \ \mathbf{A}^2\mathbf{B} \cdots \mathbf{A}^{n-1}\mathbf{B}]$$

has rank n (full row rank).

Controllability

- Definition
- Examples of controllable and uncontrollable systems
- Theorems

Example: Is the following state equation controllable

$$\dot{\mathbf{x}} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u ?$$

Controllability

- Definition
- Examples of controllable and uncontrollable systems
- Theorems

Exercise: Is the following state equation controllable

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{u} ?$$

Controllability

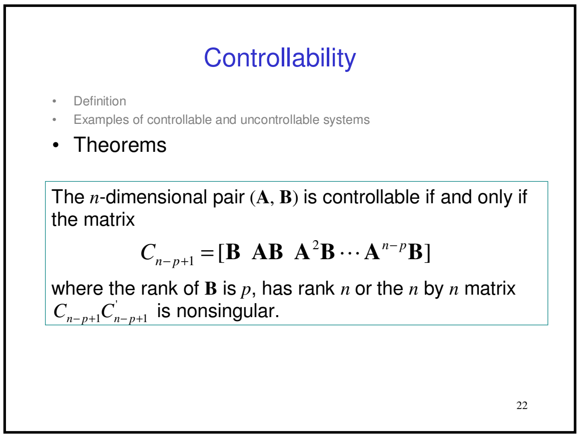
- Definition
- Examples of controllable and uncontrollable systems
- Theorems

An n by m matrix $\mathbf{H}$, with $n < m$, has rank n, if and only if the n by n matrix $\mathbf{HH'}$ has rank n or $\det(\mathbf{HH'})$ is nonzero.

The n-dimensional pair $(\mathbf{A}, \mathbf{B})$ is controllable if and only if the n by np controllability matrix

$$C = [\mathbf{B}\ \ \mathbf{AB}\ \ \mathbf{A}^2\mathbf{B} \cdots \mathbf{A}^{n-1}\mathbf{B}]$$

has rank n (full row rank) or the n by n matrix $\mathbf{CC'}$ is nonsingular.



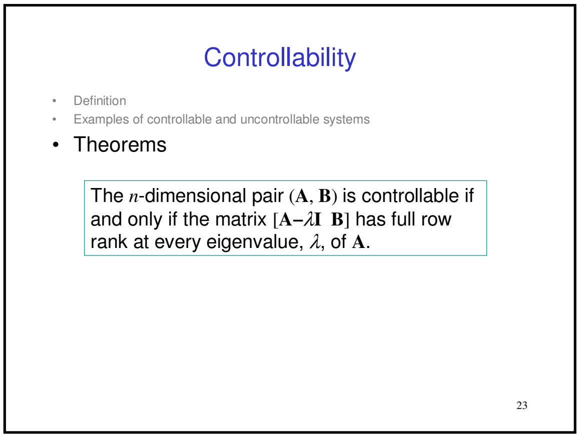
Controllability

- Definition
- Examples of controllable and uncontrollable systems
- Theorems

The n-dimensional pair $(\mathbf{A}, \mathbf{B})$ is controllable if and only if the matrix

$$C_{n-p+1} = [\mathbf{B}\ \ \mathbf{AB}\ \ \mathbf{A}^2\mathbf{B} \cdots \mathbf{A}^{n-p}\mathbf{B}]$$

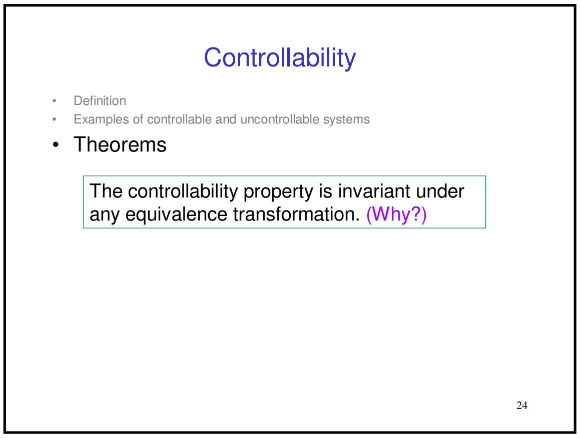
where the rank of $\mathbf{B}$ is p, has rank n or the n by n matrix $C_{n-p+1}C'_{n-p+1}$ is nonsingular.



Controllability

- Definition
- Examples of controllable and uncontrollable systems
- Theorems

The n-dimensional pair $(\mathbf{A}, \mathbf{B})$ is controllable if and only if the matrix $[\mathbf{A}-\lambda\mathbf{I}\ \ \mathbf{B}]$ has full row rank at every eigenvalue, λ, of $\mathbf{A}$.



Controllability

- Definition
- Examples of controllable and uncontrollable systems
- Theorems

The controllability property is invariant under any equivalence transformation. (Why?)

References

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