ECE 2646: Linear System Theory (3 Credits, Fall 2009)

Lecture 6: State-Space Realizations and Stability

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Outline of this lecture

- Homework 3
- Review of last lecture
- Realizations
- Definitions of stability in various domains
- Input-output stability
- Internal stability

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Homework 3

- Problems 4.11, 5.4, 5.7, 6.1 and 6.2 (for these two problems only solve the controllability parts), 6.3, and the following problem

> Suppose that $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with corresponding eigenvalue b. Please prove that $\mathbf{v}$ is also an eigenvector of the polynomial $f(\mathbf{A})$ with corresponding eigenvalue $f(b)$.

- This homework will not be graded, but solution will be provided to you next week

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Review of last lecture

- Exponential function of a square matrix

Question: Do the following equations hold?

$$e^{\mathbf{0}_{n\times n}} = 1$$
$$e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1}e^{\mathbf{A}t_2}$$
$$[e^{\mathbf{A}t}]^{-1} = e^{-\mathbf{A}t}$$
$$e^{(\mathbf{A}+\mathbf{B})t} = e^{\mathbf{A}t}e^{\mathbf{B}t}$$
$$(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$$

Question: How to calculate $e^{\mathbf{A}t}$?

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Review of last lecture

- Exponential function of a square matrix
- Solutions of LTI state equations

Question: Can you give two methods to find solutions of LTI state equations?

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Review of last lecture

- Exponential function of a square matrix
- Solutions of LTI state equations
  - Computed from time domain

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{Bu}(\tau)d\tau$$
$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{Bu}(\tau)d\tau + \mathbf{Du}(t)$$

  - Computed from frequency domain

$$\hat{\mathbf{x}}(s) = (s\mathbf{I}-\mathbf{A})^{-1}[\mathbf{x}(0) + \mathbf{B}\hat{\mathbf{u}}(s)]$$
$$\hat{\mathbf{y}}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}[\mathbf{x}(0) + \mathbf{B}\hat{\mathbf{u}}(s)] + \mathbf{D}\hat{\mathbf{u}}(s)$$

Question: Can we view the initial state as an input to the system?

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Review of last lecture

- Exponential function of a square matrix
- Solutions of LTI state equations
  - Computed from time domain
  - Computed from frequency domain
  - An exercise

Suppose $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Two 1-second experiments are performed. In the first, $\mathbf{x}(0) = [1 \;\; 1]'$ and $\mathbf{x}(1) = [4 \;\; -2]'$. In the second, $\mathbf{x}(0) = [1 \;\; 2]'$ and $\mathbf{x}(1) = [5 \;\; -2]'$.
(a) Find $\mathbf{x}(1)$ and $\mathbf{x}(2)$, given $\mathbf{x}(0) = [3 \;\; -1]'$.
(b) What can you say about the eigenvalues of $\mathbf{A}$?

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Review of last lecture

- Exponential function of a square matrix
- Solutions of LTI state equations
- Equivalence and zero-state equivalence

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Realizations

- Definitions
  - Realization problem is to find a state-space equation from a given transfer function or transfer matrix

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Realizations

- Definitions
  - Realization problem is to find a state-space equation from a given transfer function or transfer matrix
  - A transfer matrix $\hat{\mathbf{G}}(s)$ is said to be realizable if there exists a finite-dimensional state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

such that

$$\hat{\mathbf{G}}(s) = \mathbf{D} + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

and $\{\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\}$ is called a realization of $\hat{\mathbf{G}}(s)$

Question: Is the following transfer function realizable?

$$\hat{g}(s) = e^{Ts}$$

Realizations

- Definitions
- Conditions for realizability
  - A transfer matrix $\hat{\mathbf{G}}(s)$ is realizable if and only if $\hat{\mathbf{G}}(s)$ is a proper rational matrix

Realizations

- Definitions
- Conditions for realizability
- Controllable canonical form
  - Procedure to obtain the controllable canonical form

$\hat{\mathbf{G}}(s) = \hat{\mathbf{G}}(\infty) + \hat{\mathbf{G}}_{sp}(s)$,

where $\hat{\mathbf{G}}(s)$ is a $q \times p$ proper rational matrix

and $\hat{\mathbf{G}}_{sp}(s)$ is the strictly proper part of $\hat{\mathbf{G}}(s)$

↓

$$\hat{\mathbf{G}}_{sp}(s) = \frac{1}{d(s)}[\mathbf{N}(s)] = \frac{1}{s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r}[\mathbf{N}_1 s^{r-1} + \mathbf{N}_2 s^{r-2} + \cdots + \mathbf{N}_{r-1}s + \mathbf{N}_r],$$

where $d(s)$ is the least common denominator of all entries of $\hat{\mathbf{G}}_{sp}(s)$,

it is monic, i.e., with 1 as its leading coefficient,

and $\mathbf{N}_i$ are $q \times p$ constant matrices

Realizations

- Definitions
- Conditions for realizability
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where $d(s)$ is the least common denominator of all entries of $\hat{\mathbf{G}}_{sp}(s)$,
it is monic, i.e., with 1 as its leading coefficient,
and $\mathbf{N}_i$ are $q \times p$ constant matrices

↓

$$\dot{\mathbf{x}} = \begin{bmatrix} -\alpha_1\mathbf{I}_p & -\alpha_2\mathbf{I}_p & \cdots & -\alpha_{r-1}\mathbf{I}_p & -\alpha_r\mathbf{I}_p \\ \mathbf{I}_p & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_p & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_p & \mathbf{0} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \mathbf{I}_p \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} u$$

$$\mathbf{y} = [\mathbf{N}_1 \ \mathbf{N}_2 \ \cdots \ \mathbf{N}_{r-1} \ \mathbf{N}_r]\mathbf{x} + \hat{\mathbf{G}}(\infty)\mathbf{u}$$

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Realizations

- Definitions
- Conditions for realizability
- Controllable canonical form
  - Procedure to obtain controllable canonical form

Example: Find realizations of the following systems

$$\hat{\mathbf{G}}_1(s) = \frac{s^2 + s + 1}{(2s+1)(s+2)}$$

$$\hat{\mathbf{G}}_2(s) = \begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix}$$

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Realizations

- Definitions
- Conditions for realizability
- Controllable canonical form
  - Procedure to obtain controllable canonical form

Exercise: Find realizations of the following system

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \frac{4s-10}{2s+1} \\ \frac{1}{(2s+1)(s+2)} \end{bmatrix}$$

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Definitions of stability in various domains

- Ecological stability, measure of the probability of a population returning quickly to a previous state, or not going extinct
- Social stability, lack of civil unrest in a society
- Quotes: “Every time I try to define a perfectly stable person, I am appalled by the dullness of that person.”
  – J. D. Griffin

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Definitions of stability in various domains

- Ecological stability, measure of the probability of a population returning quickly to a previous state, or not going extinct
- Social stability, lack of civil unrest in a society
- Quotes: “Every time I try to define a perfectly stable person, I am appalled by the dullness of that person.” – J. D. Griffin
- In control theory, by stability, we usually mean that a stable system remains under control

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Input-output stability

- BIBO stable
  - An input $u(t)$ is said to be bounded if $u(t)$ does not grow to positive or negative infinity or, equivalently, there exists a constant u_m such that

    $$|u(t)| \le u_m < \infty \quad \text{for all } \; t \ge 0$$

  - A system is said to be BIBO stable (bounded-input bounded-output stable) if every bounded input excites a bounded output. Note that this stability is defined for the zero-state response and is applicable only if the system is initially relaxed

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Input-output stability

- BIBO stable
- Theorems

A SISO system is BIBO stable if and only if its impulse response $g(t)$ is absolutely integrable in $[0, \infty)$, or

$$\int_0^\infty |g(t)|dt \le M < \infty$$

For some constant M.

Question: Is an integrator BIBO stable? Is a differentiator BIBO stable?

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Input-output stability

- BIBO stable
- Theorems

If a system with impulse response $g(t)$ is BIBO stable, then as t approaches ∞:

1. The output excited by $u(t) = a1(t)$ approaches $\hat{g}(0) \cdot a$

2. The output excited by $u(t) = \sin \omega_0 t \, 1(t)$ approaches

$$|\hat{g}(j\omega_0)| \sin(\omega_0 t + \angle \hat{g}(j\omega_0))$$

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Input-output stability

- BIBO stable
- Theorems

A SISO system with proper rational transfer function is BIBO stable if and only if every pole of the transfer function has a negative real part or, equivalently, lies inside the left-half s-plane.

Question: Are the following systems BIBO stable?

$$\frac{1}{s^2+2}, \quad \frac{s-2}{(s+2)^2+2}, \quad \frac{(s+2)(s+1)}{s+3}.$$

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Input-output stability

- BIBO stable
- Theorems

A multivariable system with impulse response matrix $\mathbf{G}(t) = [g_{ij}(t)]$ is BIBO stable if and only if every $g_{ij}(t)$ is absolutely integrable in $[0, \infty)$.

A multivariable system with proper rational transfer-function matrix is BIBO stable if and only if every pole of every element of the transfer-function matrix has a negative real part.

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Internal stability

- Definition
  - The zero-input response of the following system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

or the equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is marginally stable or stable in the sense of Lyapunov if every finite initial state excites a bounded response. It is asymptotically stable if every finite initial state excites a bounded response, which in addition, approaches $\mathbf{0}$ as t approaches ∞

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Internal stability

- Definition

Example: An experiment about rotational stability. Consider a rigid body for which all of the principal moments of inertia are distinct. Let $I_1 > I_2 > I_3$. Suppose that the body is freely rotating about one of its principal axes. What happens when the body is slightly disturbed?

Let the body be initially rotating about principal axis 1, so that $\omega = \omega_1\mathbf{e}_1$. If we apply a slight perturbation then the angular velocity becomes $\boldsymbol{\omega} = \omega_1\mathbf{e}_1 + u\mathbf{e}_2 + v\mathbf{e}_3$, where u and v are both assumed to be small. By using Euler's equations and performing linearization, we can get the following state-space equation

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & \frac{(I_3 - I_1)\omega_1}{I_2} \\ -\frac{(I_2 - I_1)\omega_1}{I_3} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

A little more about rotational stability

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Internal stability

- Definition
- Theorems

If $\mathbf{A}$ has distinct eigenvalues, then the equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is marginally stable if and only if all eigenvalues of $\mathbf{A}$ have zero or negative real parts.

The equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is asymptotically stable if and only all eigenvalues of $\mathbf{A}$ have negative real parts.

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Internal stability

- Definition
- Theorems

Question: Does “marginally stable” implies “BIBO stable”? Does “BIBO stable” implies “marginally stable”?

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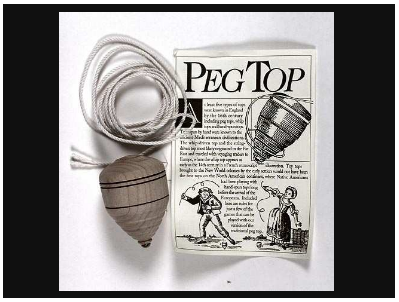

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