ECE 2646: Linear System Theory (3 Credits, Fall 2009)

Lecture 5: State-Space Solutions and Realizations

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Outline of this lecture

- Review of last lecture
- Exponential function of a square matrix
- Solution of LTI state equations
- Equivalent state equations
- Realizations

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Review of last lecture

Question: What is the geometric interpretation of determinant?

Question: What is the geometric interpretation of eigenvector and eigenvalue?

Question: How to calculate eigenvalues and eigenvectors?

Exercise: Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}.$$

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Review of last lecture

Question: Is every square matrix diagonalizable?

Question: Why do we want to diagonalize a square matrix and how do we diagonalize a matrix?

Exercise: Given

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix},$$

find A^{20}. Hint:

$$\mathbf{A} = \mathbf{Q}\hat{\mathbf{A}}\mathbf{Q}^{-1},\ \mathbf{Q} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix},\ \hat{\mathbf{A}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

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Review of last lecture

Exercise: Considering a linear transformation $\mathbf{P}$ such that

$$\bar{\mathbf{x}} = \mathbf{P}\mathbf{x},$$

what is the relationship between those matrices in the following two state-space descriptions? And for which value of $\mathbf{P}$ will the matrix $\bar{\mathbf{A}}$ be diagonalized?

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned} \quad \xrightarrow{\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}} \quad \begin{aligned} \dot{\bar{\mathbf{x}}} &= \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{B}}\mathbf{u} \\ \mathbf{y} &= \bar{\mathbf{C}}\bar{\mathbf{x}} + \bar{\mathbf{D}}\mathbf{u} \end{aligned}$$

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Diagonalization and decoupled control

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Diagonalization and decoupled control

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Diagonalization and decoupled control

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Review of last lecture

$\mathbf{A}^0 = ???$

$f(x) = x^3 + 4x^2 + 3x + 1 \Rightarrow f(\mathbf{A}) = ???$

$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_2 \end{bmatrix} \Rightarrow \mathbf{A}^k = ??? \Rightarrow f(\mathbf{A}) = ???$

Question:

$\hat{\mathbf{A}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$ or $\mathbf{A} = \mathbf{Q}\hat{\mathbf{A}}\mathbf{Q}^{-1} \Rightarrow$

$f(\hat{\mathbf{A}}) = \mathbf{Q}^{-1} f(\mathbf{A})\mathbf{Q}$ or $f(\mathbf{A}) = \mathbf{Q} f(\hat{\mathbf{A}})\mathbf{Q}^{-1}$

Why?

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Review of last lecture

Cayley-Hamilton theorem: A matrix satisfies its own characteristic polynomial

Let $\Delta(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n$
be the characteristic polynomial of A. Then
$$\Delta(\mathbf{A}) = \mathbf{A}^n + \alpha_1\mathbf{A}^{n-1} + \cdots + \alpha_{n-1}\mathbf{A} + \alpha_n\mathbf{I} = \mathbf{0}$$

Cayley-Hamilton theorem implies that, for any polynomial $f(\lambda)$, no matter how large its degree is, $f(\mathbf{A})$ can always be expressed as
$$f(\mathbf{A}) = \beta_{n-1}\mathbf{A}^{n-1} + \cdots + \beta_1\mathbf{A} + \beta_0\mathbf{I}$$
In other words, every polynomial of $\mathbf{A}$ can be expressed as a linear combination of $\{\mathbf{A}^{n-1}, \cdots, \mathbf{A}, \mathbf{I}\}$

Question: Can you prove this?

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Exponential function of a square matrix

$$e^{\lambda t} = 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \cdots + \frac{\lambda^n t^n}{n!} + \cdots$$

$$e^{\mathbf{A}t} = 1 + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \mathbf{A}^k$$

Question: What are the corresponding properties of $e^{\lambda t}$?

$e^{\mathbf{0}} = \mathbf{I}$

$e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1} e^{\mathbf{A}t_2}$

$[e^{\mathbf{A}t}]^{-1} = e^{-\mathbf{A}t}$

$\frac{d}{dt} e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$

$e^{(\mathbf{A}+\mathbf{B})t} \neq e^{\mathbf{A}t} e^{\mathbf{B}t}$ in general

$e^{(\mathbf{A}+\mathbf{B})t} = e^{\mathbf{A}t} e^{\mathbf{B}t}$ only if $\mathbf{AB} = \mathbf{BA}$

$L[e^{\mathbf{A}t}] = (s\mathbf{I} - \mathbf{A})^{-1}$

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Exponential function of a square matrix

Example: Calculate $e^{\mathbf{A}t}$ for the following matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ and}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}.$$

Reminder: $\mathbf{A}^{-1} = \frac{\text{Adj}\,\mathbf{A}}{\det \mathbf{A}} = \frac{1}{\det \mathbf{A}}[c_{ij}]'$

$$\hat{\mathbf{A}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} \Rightarrow f(\hat{\mathbf{A}}) = \mathbf{Q}^{-1} f(\mathbf{A})\mathbf{Q}$$

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Exponential function of a square matrix

Remark: Using inverse Laplace transform to calculate $e^{\mathbf{A}t}$:

$$e^{\mathbf{A}t} = L^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right]$$

Example: Calculate $e^{\mathbf{A}t}$ for $\mathbf{A} = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$

Exercise: Calculate $e^{\mathbf{A}t}$ for $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}.$

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Solution of LTI state equations

- Computed in the time domain

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \Rightarrow$$

$$e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \Rightarrow$$

$$\frac{d}{dt}[e^{-\mathbf{A}t}\mathbf{x}(t)] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \Rightarrow$$

$$e^{-\mathbf{A}\tau}\mathbf{x}(\tau)\Big|_{\tau=0}^{t} = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \Rightarrow$$

$$e^{-\mathbf{A}t}\mathbf{x}(t) - e^{\mathbf{0}}\mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \Rightarrow$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Remark: Think about the differential equation of order 1.

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Solution of LTI state equations

- Computed in the time domain

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Example: Find the solution of the following system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u(t).$$

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Solution of LTI state equations

- Computed in the time domain
- Computed in the frequency domain (using Laplace transform)

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)+\mathbf{D}\mathbf{u}(t)\end{aligned} \quad \xrightarrow{\text{Laplace transform}} \quad \begin{aligned}s\hat{\mathbf{x}}(s)-\mathbf{x}(0) &= \mathbf{A}\hat{\mathbf{x}}(s)+\mathbf{B}\hat{\mathbf{u}}(s)\\ \hat{\mathbf{y}}(s) &= \mathbf{C}\hat{\mathbf{x}}(s)+\mathbf{D}\hat{\mathbf{u}}(s)\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{x}}(s) &= (s\mathbf{I}-\mathbf{A})^{-1}[\mathbf{x}(0)+\mathbf{B}\hat{\mathbf{u}}(s)]\\ \hat{\mathbf{y}}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}[\mathbf{x}(0)+\mathbf{B}\hat{\mathbf{u}}(s)]+\mathbf{D}\hat{\mathbf{u}}(s)\end{aligned}$$

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Example: Find the solution of the following system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u(t).$$

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Solution of LTI state equations

- Computed in the time domain
- Computed in the frequency domain (using Laplace transform)

$$\begin{aligned}\hat{\mathbf{x}}(s) &= (s\mathbf{I}-\mathbf{A})^{-1}[\mathbf{x}(0)+\mathbf{B}\hat{\mathbf{u}}(s)]\\ \hat{\mathbf{y}}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}[\mathbf{x}(0)+\mathbf{B}\hat{\mathbf{u}}(s)]+\mathbf{D}\hat{\mathbf{u}}(s)\end{aligned}$$

Exercise: Show that the following system can be used as an oscillator

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{x}(t).$$

Question: Can we get oscillation from an LTI system with just a single state variable?

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Equivalent state equations

- Definition
  - Let $\mathbf{P}$ be an n by n real nonsingular matrix and let $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$. Then the state equation,

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{B}}\mathbf{u}$$
$$\mathbf{y} = \bar{\mathbf{C}}\bar{\mathbf{x}} + \bar{\mathbf{D}}\mathbf{u},$$

where

$$\bar{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1},\ \bar{\mathbf{B}} = \mathbf{P}\mathbf{B},\ \bar{\mathbf{C}} = \mathbf{C}\mathbf{P}^{-1},\ \bar{\mathbf{D}} = \mathbf{D},$$

is said to be (algebraically) equivalent to

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

and is $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ called an equivalence transformation

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Equivalent state equations

- Definition
- Properties
  - Equivalent state equations have the same characteristic polynomial and, consequently, the same set of eigenvalues and same transfer matrix (i.e. transfer-function matrix). Why?

Question: Are the following two systems equivalent to each other?

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u \quad \overset{?}{\longleftrightarrow} \quad \dot{\bar{\mathbf{x}}} = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}\bar{\mathbf{x}} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}u$$

$$\mathbf{y} = [1\ 0\ 0]\mathbf{x} \qquad \mathbf{y} = [1\ 1\ 0]\bar{\mathbf{x}}$$

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Equivalent state equations

- Definition
- Properties
- Zero-state equivalence
  - Two state equations are said to be zero-state equivalent if they have the same transfer matrix

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Realizations

- Definitions
  - Realization problem is to find a state-space equation from a given transfer function or transfer matrix

Realizations

- Definitions
  - Realization problem is to find a state-space equation from a given transfer function or transfer matrix
  - A transfer matrix $\hat{\mathbf{G}}(s)$ is said to be realizable if there exists a finite-dimensional state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

such that

$$\hat{\mathbf{G}}(s) = \mathbf{D} + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

and $\{\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\}$ is called a realization of $\hat{\mathbf{G}}(s)$

Question: Is the following transfer function realizable?

$$\hat{g}(s) = e^{Ts}$$

Realizations

- Definitions
- Conditions for realizability
  - A transfer matrix $\hat{\mathbf{G}}(s)$ is realizable if and only if $\hat{\mathbf{G}}(s)$ is a proper rational matrix

Realizations

- Definitions
- Conditions for realizability
- Controllable canonical form
  - Procedure to obtain the controllable canonical form

$$\hat{\mathbf{G}}(s) = \hat{\mathbf{G}}(\infty) + \hat{\mathbf{G}}_{sp}(s),$$
where $\hat{\mathbf{G}}(s)$ is a $q \times p$ proper rational matrix
and $\hat{\mathbf{G}}_{sp}(s)$ is the strictly proper part of $\hat{\mathbf{G}}(s)$

↓

$$\hat{\mathbf{G}}_{sp}(s) = \frac{1}{d(s)}[\mathbf{N}(s)] = \frac{1}{s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r}[\mathbf{N}_1 s^{r-1} + \mathbf{N}_2 s^{r-2} + \cdots + \mathbf{N}_{r-1}s + \mathbf{N}_r],$$
where $d(s)$ is the least common denominator of all entries of $\hat{\mathbf{G}}_{sp}(s)$,
it is monic, i.e., with 1 as its leading coefficient,
and $\mathbf{N}_i$ are $q \times p$ constant matrices

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Realizations

- Definitions
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- Controllable canonical form
  - Procedure to obtain controllable canonical form

$\hat{\mathbf{G}}(s) = \hat{\mathbf{G}}(\infty) + \hat{\mathbf{G}}_{sp}(s)$, where $\hat{\mathbf{G}}(s)$ is a $q \times p$ proper rational matrix and $\hat{\mathbf{G}}_{sp}(s)$ is the strictly proper part of $\hat{\mathbf{G}}(s)$ → $\hat{\mathbf{G}}_{sp}(s) = \frac{1}{d(s)}[\mathbf{N}(s)] = \frac{1}{s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r}[\mathbf{N}_1 s^{r-1} + \mathbf{N}_2 s^{r-2} + \cdots + \mathbf{N}_{r-1}s + \mathbf{N}_r]$, where $d(s)$ is the least common denominator of all entries of $\hat{\mathbf{G}}_{sp}(s)$, it is monic, i.e., with 1 as its leading coefficient, and $\mathbf{N}_i$ are $q \times p$ constant matrices

↓

$$\dot{\mathbf{x}} = \begin{bmatrix} -\alpha_1\mathbf{I}_p & -\alpha_2\mathbf{I}_p & \cdots & -\alpha_{r-1}\mathbf{I}_p & -\alpha_r\mathbf{I}_p \\ \mathbf{I}_p & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_p & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_p & \mathbf{0} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \mathbf{I}_p \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}u$$
$$\mathbf{y} = [\mathbf{N}_1 \ \mathbf{N}_2 \ \cdots \ \mathbf{N}_{r-1} \ \mathbf{N}_r]\mathbf{x} + \hat{\mathbf{G}}(\infty)\mathbf{u}$$

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Realizations

- Definitions
- Conditions for realizability
- Controllable canonical form
  - Procedure to obtain controllable canonical form

Example: Find realizations of the following systems

$$\hat{\mathbf{G}}_1(s) = \frac{s^2 + s + 1}{(2s+1)(s+2)}$$

$$\hat{\mathbf{G}}_2(s) = \begin{bmatrix} \dfrac{4s-10}{2s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2} \end{bmatrix}$$

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Realizations

- Definitions
- Conditions for realizability
- Controllable canonical form
  - Procedure to obtain controllable canonical form

Exercise: Find realizations of the following system

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \dfrac{4s-10}{2s+1} \\ \dfrac{1}{(2s+1)(s+2)} \end{bmatrix}$$

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References

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