

ECE 2646: Linear System Theory (3 Credits, Fall 2009)

Lecture 4: Review of Linear Algebra (III); Solution of LTI State Equations

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1

Outline of this lecture

- About homework
- Review of last lecture
- Review of linear algebra (III)
- Solution of LTI state equations

2

About homework

- Homework 2
 - Problems 3.14, 3.16, 3.29, 4.2 (use only one method), and 4.8
 - Due 10/7 (Wednesday, two weeks later) in class but before the lecture starts

3

About homework

• Homework 2

• About Homework 1

$$m_1 \frac{d^2 y_1(t)}{dt^2} = u_1(t)$$

$$m_2 \frac{d^2 y_2(t)}{dt^2} = u_2(t)$$

$$\rightarrow \begin{aligned} x_1(t) &= y_1(t), & x_2(t) &= \dot{y}_1(t) \\ x_3(t) &= y_2(t), & x_4(t) &= \dot{y}_2(t) \end{aligned}$$

Exercise: Please find state-space equations for the above system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1/m_1 & 0 \\ 0 & 0 \\ 0 & 1/m_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

4

About homework

• Homework 2

• About Homework 1

$$m\ddot{y} = -k\dot{m} - mg \rightarrow x_1 = y, \quad x_2 = \dot{y}, \quad x_3 = m, \quad u = \dot{m}$$

Question: How to linearize this system?

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \mathbf{h}(\mathbf{x}, u) = \begin{bmatrix} x_2 \\ -\frac{k}{x_3} u - g \\ u \end{bmatrix}$$

$$y = x_1$$

5

About homework

• Homework 2

• About Homework 1

$$m\ddot{y} = -k\dot{m} - mg$$

Consider an operating point at \mathbf{x}_0, u_0, y_0

$$\begin{aligned} x_1 &= y, \quad x_2 = \dot{y}, \quad x_3 = m, \quad u = \dot{m} \\ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \mathbf{h}(\mathbf{x}, u) = \begin{bmatrix} x_2 \\ -\frac{k}{x_3} u - g \\ u \end{bmatrix} \\ y &= x_1 \end{aligned}$$

Consider perturbation around the operating point
 $x_1 = x_{10} + \bar{x}_1, \quad x_2 = x_{20} + \bar{x}_2, \quad x_3 = x_{30} + \bar{x}_3$
 $u = u_0 + \bar{u}, \quad y = y_0 + \bar{y}$

$$\begin{bmatrix} \dot{x}_{10} \\ \dot{x}_{20} \\ \dot{x}_{30} \end{bmatrix} = \mathbf{h}(\mathbf{x}_0, u_0) = \begin{bmatrix} x_{20} \\ -\frac{k}{x_{30}} u_0 - g \\ u_0 \end{bmatrix}$$

$$y_0 = x_{10}$$

$$\begin{bmatrix} \dot{x}_{10} + \dot{\bar{x}}_1 \\ \dot{x}_{20} + \dot{\bar{x}}_2 \\ \dot{x}_{30} + \dot{\bar{x}}_3 \end{bmatrix} = \mathbf{h}(\mathbf{x}_0 + \bar{\mathbf{x}}, u_0 + \bar{u}) = \begin{bmatrix} x_{20} + \bar{x}_2 \\ -\frac{k}{x_{30} + \bar{x}_3} (u_0 + \bar{u}) - g \\ u_0 + \bar{u} \end{bmatrix}$$

$$y_0 + \bar{y} = x_{10} + \bar{x}_1$$

6

About homework

- Homework 2
- About Homework 1

$$\begin{bmatrix} \dot{x}_{20} \\ \dot{x}_{30} \\ y_0 \end{bmatrix} = \mathbf{h}(x_{20}, u_0) = \begin{bmatrix} -\frac{k}{x_{30}} u_0 - g \\ u_0 \\ x_{20} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_{20} + \bar{x}_2 \\ \dot{x}_{30} + \bar{x}_3 \\ y_0 + \bar{y} \end{bmatrix} = \mathbf{h}(x_{20} + \bar{x}_2, u_0 + \bar{u}) = \begin{bmatrix} -\frac{k}{x_{30} + \bar{x}_3} (u_0 + \bar{u}) - g \\ u_0 + \bar{u} \\ x_{20} + \bar{x}_2 \end{bmatrix}$$

$$y_0 + \bar{y} = x_{20} + \bar{x}_2$$

$$-\frac{k}{x_{30} + \bar{x}_3} (u_0 + \bar{u}) = -k \left(\frac{1}{x_{30}} + \frac{\bar{x}_3}{x_{30}^2} \right) (u_0 + \bar{u})$$

$$= -k \frac{u_0}{x_{30}} + k \frac{u_0 \bar{x}_3}{x_{30}^2} - k \frac{\bar{x}_3}{x_{30}} \bar{u}$$

$$\begin{bmatrix} \bar{x}_2 \\ \bar{x}_3 \\ \bar{u} \end{bmatrix} = \begin{bmatrix} \bar{x}_2 \\ k \frac{u_0}{x_{30}} \bar{x}_3 - k \frac{1}{x_{30}} \bar{u} \\ \bar{u} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & k \frac{u_0}{x_{30}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -k \frac{1}{x_{30}} \\ 1 \end{bmatrix} \bar{u}$$

$$\bar{y} = \bar{x}_1 = [1 \ 0 \ 0] \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix}$$

Review of last lecture

- Rational transfer functions (still remember these concepts: proper, strictly proper, biproper, improper, pole, zero, and coprime?)

Question: Why improper rational transfer functions rarely arise in practice?

8

Review of last lecture

- Rational transfer functions
- State-space equations

Question: What is the transfer function of

$$m \frac{d^2 y(t)}{dt^2} = u(t)$$

under initial condition $y(0) = 1, \dot{y}(0) = 2$?

9

Review of last lecture

- Rational transfer functions

- State-space equations

Exercise: Please derive the transfer-function matrix for the following system.

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned}$$

10

Review of last lecture

- Rational transfer functions
- State-space equations

- Linearization

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t), t) \\ \dot{\mathbf{x}}_0(t) &= \mathbf{h}(\mathbf{x}_0(t), \mathbf{u}_0(t), t) \\ \mathbf{x}(t) &= \mathbf{x}_0(t) + \bar{\mathbf{x}}(t) \\ \mathbf{u}(t) &= \mathbf{u}_0(t) + \bar{\mathbf{u}}(t)\end{aligned}$$

→

$$\begin{aligned}\dot{\mathbf{x}}_0(t) + \dot{\bar{\mathbf{x}}}(t) &= \mathbf{h}(\mathbf{x}_0(t) + \bar{\mathbf{x}}(t), \mathbf{u}_0(t) + \bar{\mathbf{u}}(t), t) \\ &= \mathbf{h}(\mathbf{x}_0(t), \mathbf{u}_0(t), t) + \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \bar{\mathbf{x}} + \frac{\partial \mathbf{h}}{\partial \mathbf{u}} \bar{\mathbf{u}} + \dots\end{aligned}$$

$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{A}(t)\bar{\mathbf{x}}(t) + \mathbf{B}(t)\bar{\mathbf{u}}$$

$$\mathbf{h} = [h_1 \dots h_n], \mathbf{x} = [x_1 \dots x_n], \mathbf{u} = [u_1 \dots u_p]$$

$$\mathbf{A}(t) = \frac{\partial \mathbf{h}}{\partial \mathbf{x}} = \begin{bmatrix} \partial h_1 / \partial x_1 & \partial h_1 / \partial x_2 & \dots & \partial h_1 / \partial x_n \\ \partial h_2 / \partial x_1 & \partial h_2 / \partial x_2 & \dots & \partial h_2 / \partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial h_n / \partial x_1 & \partial h_n / \partial x_2 & \dots & \partial h_n / \partial x_n \end{bmatrix}$$

$$\mathbf{B}(t) = \frac{\partial \mathbf{h}}{\partial \mathbf{u}} = \begin{bmatrix} \partial h_1 / \partial u_1 & \partial h_1 / \partial u_2 & \dots & \partial h_1 / \partial u_p \\ \partial h_2 / \partial u_1 & \partial h_2 / \partial u_2 & \dots & \partial h_2 / \partial u_p \\ \vdots & \vdots & \ddots & \vdots \\ \partial h_n / \partial u_1 & \partial h_n / \partial u_2 & \dots & \partial h_n / \partial u_p \end{bmatrix}$$

Jacobians, computed at $\mathbf{x}_0(t), \mathbf{u}_0(t)$

11

Review of last lecture

- Rational transfer functions
- State-space equations

- Linearization

Exercise: Please use Jacobians to linearize the following system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \mathbf{h}(\mathbf{x}, u) = \begin{bmatrix} x_2 \\ -\frac{k}{x_3}u - g \\ u \end{bmatrix}$$

$y = x_1$

12

Review of linear algebra (III)

- Determinant
 - Determinant is a scalar defined for a square matrix

If two rows (columns) are switched, the determinant changes the sign.
If a whole row (column) is scaled by a number k , the determinant is scaled by a number k .
If an entire row or an entire column is 0, the determinant is 0.

Question1: Does $\det k\mathbf{A}$ equal $k \det \mathbf{A}$?

Question2: Does the operation \det constitute a linear system?

13

Review of linear algebra (III)

- Determinant
 - Determinant is a scalar defined for a square matrix
 - Determinant of a triangular matrix or block triangular matrix

$$\det \begin{bmatrix} A_1 & * & * & * \\ 0 & A_2 & * & * \\ 0 & 0 & A_3 & * \\ 0 & 0 & 0 & A_4 \end{bmatrix} = ?$$

If \mathbf{A} is a square block upper triangular matrix, in which the diagonal blocks are square, then the determinant of \mathbf{A} equals the product of the determinants of its diagonal blocks.

14

Review of linear algebra (III)

- Determinant
 - Determinant is a scalar defined for a square matrix
 - Determinant of a triangular matrix
 - Elementary operations that preserve the determinant
 - Determinant of product of matrices

$$\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$$

(or $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$)

Question: How about $\det(\mathbf{A} + \mathbf{B}) = \det \mathbf{A} + \det \mathbf{B}$?

15

Review of linear algebra (III)

- Determinant

- Determinant is a scalar defined for a square matrix
- Determinant of a triangular matrix
- Elementary operations that preserve the determinant
- Determinant of product of matrices

- Laplace expansion

$$\det \mathbf{A} = \sum_{i=1}^n a_{ij} c_{ij},$$

where a_{ij} denotes the entry at the i -th row and j -th column of \mathbf{A} , the number c_{ij} is the cofactor corresponding to a_{ij} and equals $(-1)^{i+j} \det M_{ij}$, and M_{ij} is the $(n-1)$ by $(n-1)$ submatrix of \mathbf{A} by deleting its i -th row and j -th column.

16

Review of linear algebra (III)

- Determinant

- Determinant is a scalar defined for a square matrix
- Determinant of a triangular matrix
- Elementary operations that preserve the determinant
- Determinant of product of matrices

- Laplace expansion

Exercise:

$$\begin{vmatrix} 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 2 & 3 \\ -1 & 1 & 3 & 1 & 2 \\ 2 & 2 & 0 & 0 & 3 \end{vmatrix}.$$

17

Review of linear algebra (III)

- Determinant

- Determinant is a scalar defined for a square matrix
- Determinant of a triangular matrix
- Elementary operations that preserve the determinant
- Determinant of product of matrices
- Laplace expansion

- Inverse of square matrices

$$\mathbf{A}^{-1} = \frac{\text{Adj } \mathbf{A}}{\det \mathbf{A}} = \frac{1}{\det \mathbf{A}} [c_{ij}]',$$

where c_{ij} is the cofactor corresponding to a_{ij} (the entry at the i -th row and j -th column of \mathbf{A}) and equals $(-1)^{i+j} \det M_{ij}$, and M_{ij} is the $(n-1)$ by $(n-1)$ submatrix of \mathbf{A} by deleting its i -th row and j -th column.

18

Review of linear algebra (III)

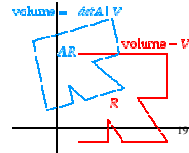
- Determinant

- Determinant is a scalar defined for a square matrix
- Determinant of a triangular matrix
- Elementary operations that preserve the determinant
- Determinant of product of matrices
- Laplace expansion
- Inverse of square matrices

- Geometry of determinant

- Determinant indicates the ratio for the change of volume under the transformation corresponding to the matrix

Let A be an n by n matrix. Let R be a region in \mathbb{R}^n , and AR be the image under the transformation given by A . Then $(\text{volume of } AR) = |\det A| (\text{volume of } R)$.



Review of linear algebra (III)

- Determinant

Remark: For an n by n matrix A , the following statements are equivalent to each other:

- (1) A is nonsingular;
- (2) A is invertible;
- (3) A has full rank;
- (4) all the columns (rows) of A are linearly independent;
- (5) $\det A$ is nonzero;
- (6) $Ax = 0$ does not have nonzero solutions.

Review of linear algebra (III)

- Determinant

- Eigenvalue and eigenvector

- Definition

- A real or complex number λ is called an **eigenvalue** of a square real matrix A if there exists a nonzero vector x such that $Ax = \lambda x$ (a story about eigenvalue and a friend of mine)
- Any nonzero vector x satisfying $Ax = \lambda x$ is called an **eigenvector** of A associated with eigenvalue λ

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
 - Definition
 - Characteristic polynomial of \mathbf{A}
 - Defined by $\Delta(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})$
 - Every root of the characteristic polynomial of \mathbf{A} is an eigenvalue of \mathbf{A} (why?)

22

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
 - Definition
 - Characteristic polynomial of \mathbf{A}
 - Defined by $\Delta(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})$
 - Every root of the characteristic polynomial of \mathbf{A} is an eigenvalue of \mathbf{A}

Exercise: Prove that $\det \mathbf{A}$ equals product of all eigenvalues of \mathbf{A} .

23

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
 - Definition
 - Characteristic polynomial of \mathbf{A}
 - Defined by $\Delta(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})$
 - Every root of the characteristic polynomial of \mathbf{A} is an eigenvalue of \mathbf{A}

Exercise: What are the eigenvalues of the following matrix (a matrix to rotate a column vector in Cartesian coordinates about the origin by an anti-clockwise angle of θ)?

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

24

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
 - Definition
 - Characteristic polynomial of A
 - Defined by $\Delta(\lambda) = \det(\lambda I - A)$
 - Every root of the characteristic polynomial of A is an eigenvalue of A

Remark: For an n by n matrix A , the following statements are equivalent to each other:

- (1) A is nonsingular;
- (2) A is invertible;
- (3) A has full rank;
- (4) all the columns (rows) of A are linearly independent;
- (5) $\det A$ is nonzero;
- (6) $Ax = 0$ does not have nonzero solutions;
- (7) A does not have zero eigenvalue.

25

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
 - Definition
 - Characteristic polynomial of A
 - Companion-form matrices

Exercise: What are the characteristic polynomial of the following matrices?

$$\begin{bmatrix} 0 & 0 & 0 & -a_4 \\ 1 & 0 & 0 & -a_3 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_1 \end{bmatrix} \quad \begin{bmatrix} -a_1 - a_2 - a_3 - a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

26

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
 - Definition
 - Characteristic polynomial of A
 - Companion-form matrices

Exercise: What are the characteristic polynomial of the following matrices?

$$\begin{bmatrix} 0 & 0 & 0 & -a_4 \\ 1 & 0 & 0 & -a_3 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_1 \end{bmatrix} \quad \begin{bmatrix} -a_1 - a_2 - a_3 - a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

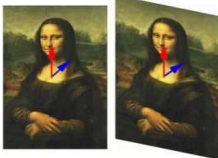
Answer:
 $\Delta(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$

Remark: These matrices (and their transposes) can easily be formed from the coefficients of $\Delta(\lambda)$ and are called companion-form matrices.

27

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
 - Definition
 - Characteristic polynomial of A
 - Companion-form matrices
 - Geometry of eigenvector
 - An eigenvector of a transformation is a non-null vector whose direction is unchanged by that transformation



Question1: Which vector (red or blue) is an eigenvector of the transformation?

Question2: What are the eigenvectors of the rotational transformation?

28

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
 - Definition
 - Characteristic polynomial of A
 - Companion-form matrices
 - Geometry of eigenvector
 - About "eigen"
 - "Eigen" was introduced from German by Hilbert to denote eigenvalues and eigenvectors, and can be translated as "own", "peculiar to", "characteristic", or "individual", emphasizing how important eigenvalues are to defining the unique nature of a specific transformation

29

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
 - Definition
 - Characteristic polynomial of A
 - Companion-form matrices
 - Geometry of eigenvector
 - About "eigen"
 - "Eigen" was introduced from German by Hilbert to denote eigenvalues and eigenvectors, and can be translated as "own", "peculiar to", "characteristic", or "individual", emphasizing how important eigenvalues are to defining the unique nature of a specific transformation
 - It is common to prefix any natural name for the solution with eigen instead of saying eigenvector. For example, eigenfunction if the eigenvector is a function, eigenmode if the eigenvector is a harmonic mode, eigenstate if the eigenvector is a quantum state, etc.

30

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
 - Definition
 - Characteristic polynomial of A
 - Companion-form matrices
 - Geometry of eigenvector
 - About "eigen"

$$T(\mathbf{v}_\lambda) = \lambda \mathbf{v}_\lambda$$

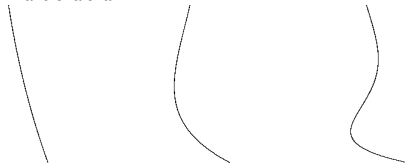
Remark: T can be any transformation, including linear transformation (e.g. a matrix or differential operator).

Question: If T is a differential operator d/dt , the eigenvectors are commonly called **eigenfunctions**. What is the solution to $T(f) = \lambda f$?

31

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
 - Definition
 - Characteristic polynomial of A
 - Companion-form matrices
 - Geometry of eigenvector
 - About "eigen"
 - A simple experiment: **eigenmodes** of a compound pendulum and of a chain



33

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
- Basis and representation
 - A set of linearly independent vectors in \mathbf{R}^n is called a **basis** if every vectors in \mathbf{R}^n can be expressed as a unique linear combination of the set
 - Any set of n linearly independent vectors can be used as a basis of \mathbf{R}^n

33

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
- **Basis and representation**
 - A set of linearly independent vectors in \mathbb{R}^n is called a basis if every vectors in \mathbb{R}^n can be expressed as a unique linear combination of the set
 - Any set of n linearly independent vectors can be used as a basis of \mathbb{R}^n
 - Let $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$ be a basis, and then every vector \mathbf{x} can be expressed uniquely as

$$\mathbf{x} = \alpha_1 \mathbf{q}_1 + \alpha_2 \mathbf{q}_2 + \dots + \alpha_n \mathbf{q}_n$$

$$= [\mathbf{q}_1 \ \mathbf{q}_2 \ \dots \ \mathbf{q}_n] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \equiv \mathbf{Q}\bar{\alpha}$$

We call $[\alpha_1, \alpha_2, \dots, \alpha_n]$ the **representation** of the vector \mathbf{x} with respect to the basis $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$

34

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
- Basis and representation
- **Diagonal form**
 - Every matrix with distinct eigenvalues can be diagonalized
 - Let $\lambda_i, i = 1, \dots, n$, be the eigenvalues of \mathbf{A} and be all distinct
 - Let \mathbf{q}_i be an eigenvector of \mathbf{A} associated with λ_i ; $\mathbf{A}\mathbf{q}_i = \lambda_i \mathbf{q}_i$
 - Then the set of eigenvectors $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$ is linearly independent (*why?*)

35

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
- Basis and representation
- **Diagonal form**
 - Every matrix with distinct eigenvalues can be diagonalized
 - Let $\lambda_i, i = 1, \dots, n$, be the eigenvalues of \mathbf{A} and be all distinct
 - Let \mathbf{q}_i be an eigenvector of \mathbf{A} associated with λ_i ; $\mathbf{A}\mathbf{q}_i = \lambda_i \mathbf{q}_i$
 - Then the set of eigenvectors $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$ is linearly independent

$$\mathbf{A}[\mathbf{q}_1 \ \mathbf{q}_2 \ \dots \ \mathbf{q}_n] = [\mathbf{q}_1 \ \mathbf{q}_2 \ \dots \ \mathbf{q}_n] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

$$\mathbf{A}\mathbf{Q} = \mathbf{Q}\hat{\mathbf{A}}$$

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \hat{\mathbf{A}}$$

36

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
- Basis and representation

• Diagonal form

- Every matrix with distinct eigenvalues can be diagonalized

$$A[q_1, q_2, \dots, q_n] = [q_1, q_2, \dots, q_n] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

$$AQ = Q\hat{\Lambda}$$

$$Q^{-1}AQ = \hat{\Lambda}$$

MATLAB command to find the eigenvalues and eigenvectors of a matrix:
`[Q, D] = eig(A)`

Question: Is the matrix Q unique?

Exercise: Prove that $\det A$ equals product of all eigenvalues of A (this also holds for the case where eigenvalues of A are not all distinct).

37

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
- Basis and representation

• Diagonal form

- Every matrix with distinct eigenvalues can be diagonalized

$$A[q_1, q_2, \dots, q_n] = [q_1, q_2, \dots, q_n] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

$$AQ = Q\hat{\Lambda}$$

$$Q^{-1}AQ = \hat{\Lambda}$$

Example: Consider the equation $Ax = y$ and linear transformation Q such that $x = Q\bar{x}$ and $y = Q\bar{y}$, then
 $Ax = y \Rightarrow$
 $AQ\bar{x} = Q\bar{y} \Rightarrow$
 $Q^{-1}AQ\bar{x} = \bar{y} \Rightarrow$
 $\hat{A}\bar{x} = \bar{y}$

38

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
- Basis and representation

• Diagonal form

- Every matrix with distinct eigenvalues can be diagonalized

$$A[q_1, q_2, \dots, q_n] = [q_1, q_2, \dots, q_n] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

$$AQ = Q\hat{\Lambda}$$

$$Q^{-1}AQ = \hat{\Lambda}$$

Exercise: Considering a linear transformation Q such that $x = Q\bar{x}$, what will the following state equations change to?

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$

$$y = C\bar{x} + D\bar{u}$$

39

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
- Basis and representation
- **Diagonal form**
 - Every matrix with distinct eigenvalues can be diagonalized
 - If a square matrix has repeated eigenvalues, then it may **not** be diagonalized; however, it has a block-diagonal and triangular-form representation (**Jordan form**)

40

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
- Basis and representation
- Diagonal form
- **Functions of a square matrix**
 - Polynomials of a square matrix **A**

$$\mathbf{A}^k \equiv \mathbf{A}\mathbf{A}\cdots\mathbf{A} \text{ (} k \text{ terms)}$$

$$\mathbf{A}^0 \equiv \mathbf{I} \text{ (identity matrix)}$$

$$f(x) = x^3 + 4x^2 + 3x + 1 \Rightarrow f(\mathbf{A}) = \mathbf{A}^3 + 4\mathbf{A}^2 + 3\mathbf{A} + \mathbf{I}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_2 \end{bmatrix} \Rightarrow \mathbf{A}^k = \begin{bmatrix} \mathbf{A}_1^k & 0 \\ 0 & \mathbf{A}_2^k \end{bmatrix} \Rightarrow f(\mathbf{A}) = \begin{bmatrix} f(\mathbf{A}_1) & 0 \\ 0 & f(\mathbf{A}_2) \end{bmatrix}$$

41

Review of linear algebra (III)

- Determinant
- Eigenvalue and eigenvector
- Basis and representation
- Diagonal form
- **Functions of a square matrix**
 - Polynomials of a square matrix **A**

$$\hat{\mathbf{A}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} \text{ or } \mathbf{A} = \mathbf{Q}\hat{\mathbf{A}}\mathbf{Q}^{-1} \Rightarrow$$

$$f(\hat{\mathbf{A}}) = \mathbf{Q}^{-1}f(\mathbf{A})\mathbf{Q} \text{ or } f(\mathbf{A}) = \mathbf{Q}f(\hat{\mathbf{A}})\mathbf{Q}^{-1}$$

Why?

42

Review of linear algebra (III)

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Exercise: Given
$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix},$$
 find \mathbf{A}^{20} . Hint:

$$\mathbf{A} = \mathbf{Q}\hat{\mathbf{A}}\mathbf{Q}^{-1}, \mathbf{Q} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \hat{\mathbf{A}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

43

Review of linear algebra (III)

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- Functions of a square matrix
 - Polynomials of a square matrix **A**
 - Cayley-Hamilton theorem: A matrix satisfies its own characteristic polynomial

Let
$$\Delta(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^n + \alpha_1\lambda^{n-1} + \dots + \alpha_{n-1}\lambda + \alpha_n$$
 be the characteristic polynomial of **A**. Then
$$\Delta(\mathbf{A}) = \mathbf{A}^n + \alpha_1\mathbf{A}^{n-1} + \dots + \alpha_{n-1}\mathbf{A} + \alpha_n\mathbf{I} = \mathbf{0}$$

44

Review of linear algebra (III)

- Determinant
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- Functions of a square matrix
 - Polynomials of a square matrix **A**
 - Cayley-Hamilton theorem: A matrix satisfies its own characteristic polynomial
 - Cayley-Hamilton theorem implies that, for any polynomial $f(\lambda)$, no matter how large its degree is, $f(\mathbf{A})$ can always be expressed as

$$f(\mathbf{A}) = \beta_{n-1}\mathbf{A}^{n-1} + \dots + \beta_1\mathbf{A} + \beta_0\mathbf{I}$$

In other words, every polynomial of **A** can be expressed as a linear combination of $\{\mathbf{A}^{n-1}, \dots, \mathbf{A}, \mathbf{I}\}$

Question: Can you prove this?

45

Review of linear algebra (III)

- Determinant
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• Functions of a square matrix

– Polynomials of a square matrix A

– Exponential function e^{At}

$$e^{\lambda t} = 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \dots + \frac{\lambda^n t^n}{n!} + \dots$$

$$e^{At} = 1 + tA + \frac{t^2}{2!} A^2 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k$$

46

Review of linear algebra (III)

- Determinant
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• Functions of a square matrix

– Polynomials of a square matrix A

– Exponential function e^{At}

$$e^{At} = 1 + tA + \frac{t^2}{2!} A^2 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k$$

$$\frac{d}{dt} e^{At} = A e^{At} = e^{At} A$$

$$e^0 = \mathbf{I}$$

$$e^{A(t_1+t_2)} = e^{At_1} e^{At_2}$$

$$[e^{At}]^{-1} = e^{-At}$$

$$e^{(A+B)t} \neq e^{At} e^{Bt} \text{ in general}$$

$$e^{(A+B)t} = e^{At} e^{Bt} \text{ only if } \mathbf{AB} = \mathbf{BA}$$

$$L[e^{At}] = (s\mathbf{I} - \mathbf{A})^{-1}$$

47

Review of linear algebra (III)

- Determinant
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- Diagonal form

• Functions of a square matrix

– Polynomials of a square matrix A

– Exponential function e^{At}

Exercise: Calculate e^{At} for the following matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ and}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

48

Solution of LTI state equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \Rightarrow$$

$$e^{-\Lambda t} \dot{\mathbf{x}}(t) - e^{-\Lambda t} \mathbf{A}\mathbf{x}(t) = e^{-\Lambda t} \mathbf{B}\mathbf{u}(t) \Rightarrow$$

$$\frac{d}{dt} [e^{-\Lambda t} \mathbf{x}(t)] = e^{-\Lambda t} \mathbf{B}\mathbf{u}(t) \Rightarrow$$

$$e^{-\Lambda \tau} \mathbf{x}(\tau) \Big|_{\tau=0}^t = \int_0^t e^{-\Lambda \tau} \mathbf{B}\mathbf{u}(\tau) d\tau \Rightarrow$$

$$e^{-\Lambda t} \mathbf{x}(t) - e^0 \mathbf{x}(0) = \int_0^t e^{-\Lambda \tau} \mathbf{B}\mathbf{u}(\tau) d\tau \Rightarrow$$

$$\mathbf{x}(t) = e^{\Lambda t} \mathbf{x}(0) + \int_0^t e^{\Lambda(t-\tau)} \mathbf{B}\mathbf{u}(\tau) d\tau$$

49

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50
