ECE 2646: Linear System Theory (3 Credits, Fall 2009)

Lecture 3: Rational Transfer Functions and State-space Equations; Linearization; Review of Linear Algebra (II)

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Instructor: Zhi-Hong Mao
Assistant Professor of ECE and Bioengineering
University of Pittsburgh, Pittsburgh, PA

Outline of this lecture

- Review of last lecture
- Rational transfer functions and state-space equations
- Linearization
- Review of linear algebra (II)

Review of last lecture

- Categories of systems
  - SISO, MIMO, SIMO, and MISO systems
  - Continuous-time system, discrete-time system, and hybrid system
  - Memoryless system and system that has memory

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  - Memoryless system and system that has memory
  - Causal system and noncausal system

Remark: Causality describes the relationship between causes and effects, and is fundamental to all natural science, especially physics. In classical physics, the cause simply had to precede its effect; in modern physics, the effect must belong to the future light cone of its cause, even if the spacetime is curved. Causality is also studied from the perspectives of philosophy, computer science, statistics, and psychology.

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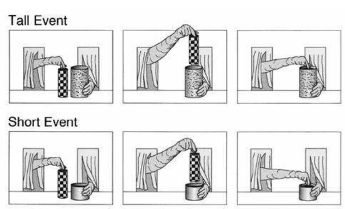
Infant causal perception



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Review of last lecture

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Question: Filtering can be classified into three different problems: estimation, prediction, and smoothing. Estimation provides an estimate of the input based upon all data up to the current time. Prediction provides an estimate of what the value ahead of the observed data. Smoothing provides an estimate using observations that are ahead of in time of the estimate. Which one of these can be realized by a physical system (in real time)?

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Question: Consider a system modeled as a map from $u(t)$ to $y(t)$, where

$$y(t) = \int_{-\infty}^{+\infty} e^{-|t-\tau|} u(\tau) d\tau.$$

Is this system causal?

Review of last lecture

- Categories of systems
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  - Memoryless system and system that has memory
  - Causal system and noncausal system
    - In engineering, a system is called a causal system if its current output depends on past and current but not future input
    - Physical systems are all causal systems, but noncausal systems did exist in movies and science fictions

My list:
Superman (1978), The Terminator (1984),
Back to Future (1985), Twelve Monkeys (1995),
Frequency (2000), The Butterfly Effect (2004), etc.

Review of last lecture

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  - Concept of state
    - The concept of state was proposed to address the following question: Given the input at some time, how much information do we need about past inputs in order to determine the present output?
      - This question addresses the issue of memory in the system
      - The answer gives us an idea of the complexity, or number of degrees of freedom, associated with the dynamic behavior of the system
      - In a control application, the answer to the above question suggests the required degree of complexity of the controller

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    - The concept of state was proposed to address the following question: Given the input at some time, how much information do we need about past inputs in order to determine the present output?
    - The state $\mathbf{x}(t_0)$ of a system at time t_0 is the information at t_0 that, together with the input $\mathbf{u}(t)$, for $t \geq t_0$, determines uniquely the output $\mathbf{y}(t)$ for all $t \geq t_0$

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    - State is not unique; length of state is not unique
      - In the example of $F = ma$, you can add to the set of state variables (position and velocity) the temperature, your age, the number of students in our class, etc.—the choice of state can be very subjective. For this case, you could also have used all atomic velocities and positions as state variables.… We will see in this course how to determine whether a set of state variables is "just enough" or redundant

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    - State is not unique; length of state is not unique
    - In the text book by Kuo (1995), the state variables of a system are defined as a **minimal set** of variables such that knowledge of these variables at any time t_0, and information on the input excitation subsequently applied, are sufficient to determine the state of the system at any time $t \geq t_0$

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Question: How many state variables are needed to describe the following systems:

(1) $y(t) = u(t) + 2$

(2) $dy(t) / dt = u(t)$ (in general, how about a n-th order linear ODE?)

(3) $y(t) = u(t - 1)$?

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Question: How can the following system be approximated by a system with finite state variables: $y(t) = u(t-1)$?

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  - Concept of state
  - Lumped system and distributed system

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Review of last lecture

- Categories of systems
- Linear systems
  - Superposition, additivity, and homogeneity

$$\left.\begin{array}{l}\mathbf{x}_1(t_0)+\mathbf{x}_2(t_0)\\ \mathbf{u}_1(t)+\mathbf{u}_2(t),\ t\ge t_0\end{array}\right\}\rightarrow \mathbf{y}_1(t)+\mathbf{y}_2(t),\ t\ge t_0$$

Additivity

$$\left.\begin{array}{l}\alpha\,\mathbf{x}_1(t_0)\\ \alpha\,\mathbf{u}_1(t),\ t\ge t_0\end{array}\right\}\rightarrow \alpha\,\mathbf{y}_1(t),\ t\ge t_0$$

Homogeneity

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Review of last lecture

- Categories of systems
- Linear systems
  - Superposition, additivity, and homogeneity

Question: Consider a system modeled as a map from $u(t)$ to $y(t)$, where

$$y(t)=\int_{-\infty}^{+\infty} e^{-|t-\tau|}u(\tau)d\tau.$$

Is this a linear system?

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Review of last lecture

- Categories of systems
- Linear systems
- LTI systems
  - In an LTI system, if the initial state is shifted to time t_0+T and the same input waveform is applied from t_0+T, the output waveform will be the same except that it starts to appear from time t_0+T

Question: Consider a system modeled as a map from $u(t)$ to $y(t)$, where

$$y(t)=\int_{-\infty}^{+\infty} e^{-|t-\tau|}u(\tau)d\tau.$$

Is this an LTI system?

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Review of last lecture

- Categories of systems
- Linear systems
- LTI systems
  - In an LTI system, if the initial state is shifted to time t_0+T and the same input waveform is applied from t_0+T, the output waveform will be the same except that it starts to appear from time t_0+T
  - Input-output description

For a causal, SISO, LTI system relaxed at 0:

The impulse response $g(t) = 0$ for $t < 0$

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

Convolution integral

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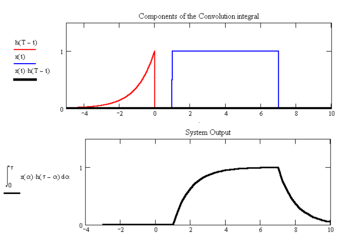
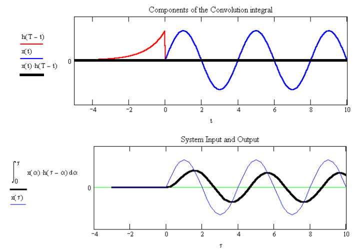
- Linear systems
- LTI systems
  - In an LTI system, if the initial state is shifted to time t_0+T and the same input waveform is applied from t_0+T, the output waveform will be the same except that it starts to appear from time t_0+T
  - Input-output description
  - A little more about convolution





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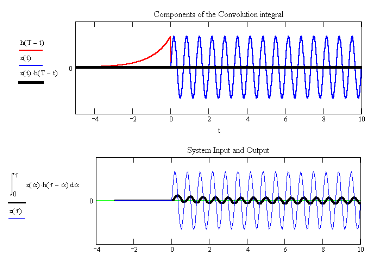

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Rational transfer functions and state-space equations

- If an LTI system is lumped, then its transfer function is a rational function of s: $\boxed{\hat{g}(s) = N(s)/D(s)}$
  - $\hat{g}(s)$ is proper, if deg (degree of) $D(s) \geq \deg N(s)$
  - $\hat{g}(s)$ is strictly proper if $\deg D(s) > \deg N(s)$
  - $\hat{g}(s)$ is biproper $\deg D(s) = \deg N(s)$
  - $\hat{g}(s)$ is improper if $\deg D(s) < \deg N(s)$

Question: Improper rational transfer functions rarely arise in practice, since they will amplify high-frequency noise. Why?

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Rational transfer functions and state-space equations

- If an LTI system is lumped, then its transfer function is a rational function of s:

$$\boxed{\hat{g}(s) = N(s)/D(s)}$$

- Poles and zeros
  - A real or complex number λ is called a pole if $D(\lambda) = 0$, and λ is called a zero if $N(\lambda) = 0$
  - If $N(s)$ and $D(s)$ have no common factors of degree 1 or higher, they are called coprime

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Rational transfer functions and state-space equations

- If an LTI system is lumped, then its transfer function is a rational function of s:

$$\boxed{\hat{g}(s) = N(s)/D(s)}$$

- Poles and zeros
- State-space equations

Exercise: Find state-space description for
$$\ddot{y} + 2\dot{y} - 3y = \dot{u} - u$$

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Rational transfer functions and state-space equations

- If an LTI system is lumped, then its transfer function is a rational function of s:

$$\boxed{\hat{g}(s) = N(s)/D(s)}$$

- Poles and zeros
- State-space equations
  - Deriving transfer-function matrix from state-space equation

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned} \xrightarrow{\text{Laplace transform}} \begin{aligned}s\hat{\mathbf{x}}(s) - \mathbf{x}(0) &= \mathbf{A}\hat{\mathbf{x}}(s) + \mathbf{B}\hat{\mathbf{u}}(s)\\ \hat{\mathbf{y}}(s) &= \mathbf{C}\hat{\mathbf{x}}(s) + \mathbf{D}\hat{\mathbf{u}}(s)\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{x}}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{u}}(s)\\ \hat{\mathbf{y}}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{u}}(s) + \mathbf{D}\hat{\mathbf{u}}(s)\end{aligned}$$

For zero initial state

$$\hat{\mathbf{y}}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\hat{\mathbf{u}}(s) \longrightarrow \hat{\mathbf{G}}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

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Linearization

- Many physical systems can be described by nonlinear differential equations

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{h}(\mathbf{x}(t),\mathbf{u}(t),t)\\ \mathbf{y}(t) &= \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t),t)\end{aligned}$$

- Some nonlinear equations can be approximated by linear equations (but how?)

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Linearization

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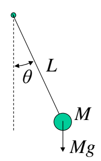
- Some nonlinear equations can be approximated by linear equations



$$ML\frac{d^2\theta(t)}{dt^2} + Mg\sin\theta(t) = 0$$

For small value of θ

$$L\frac{d^2\theta(t)}{dt^2} + g\theta(t) = 0$$

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Linearization

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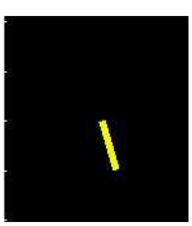
- Some nonlinear equations can be approximated by linear equations



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Linearization

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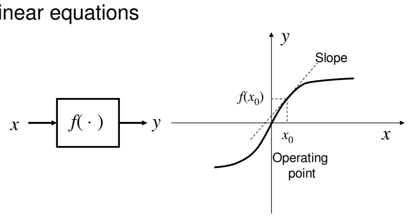

Linearization

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$$\dot{\mathbf{x}}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t), t)$$
$$\mathbf{y}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

- Some nonlinear equations can be approximated by linear equations
  - For some input $\mathbf{u}_0(t)$ and some initial state, $\mathbf{x}_0(t)$ is the solution of the above equations

$$\boxed{\dot{\mathbf{x}}_0(t) = \mathbf{h}(\mathbf{x}_0(t), \mathbf{u}_0(t), t)}$$

  - Suppose $\mathbf{x}(t) = \mathbf{x}_0(t) + \bar{\mathbf{x}}(t)$ for slightly perturbed input $\mathbf{u}(t) = \mathbf{u}_0(t) + \bar{\mathbf{u}}(t)$ and initial state

Linearization

- Many physical systems can be described by nonlinear differential equations
- Some nonlinear equations can be approximated by linear equations

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t), t) \\
\dot{\mathbf{x}}_0(t) &= \mathbf{h}(\mathbf{x}_0(t), \mathbf{u}_0(t), t) \\
\mathbf{x}(t) &= \mathbf{x}_0(t) + \bar{\mathbf{x}}(t) \\
\mathbf{u}(t) &= \mathbf{u}_0(t) + \bar{\mathbf{u}}(t)
\end{aligned}$$

$$\begin{aligned}
\dot{\mathbf{x}}_0(t) + \dot{\bar{\mathbf{x}}}(t) &= \mathbf{h}(\mathbf{x}_0(t) + \bar{\mathbf{x}}(t), \mathbf{u}_0(t) + \bar{\mathbf{u}}(t), t) \\
&= \mathbf{h}(\mathbf{x}_0(t), \mathbf{u}_0(t), t) + \frac{\partial \mathbf{h}}{\partial \mathbf{x}}\bar{\mathbf{x}} + \frac{\partial \mathbf{h}}{\partial \mathbf{u}}\bar{\mathbf{u}} + \cdots
\end{aligned}$$

$\mathbf{h} = [h_1 \cdots h_n]'$, $\mathbf{x} = [x_1 \cdots x_n]'$, $\mathbf{u} = [u_1 \cdots u_p]'$

$$\mathbf{A}(t) \equiv \frac{\partial \mathbf{h}}{\partial \mathbf{x}} = \begin{bmatrix} \partial h_1/\partial x_1 & \partial h_1/\partial x_2 & \cdots & \partial h_1/\partial x_n \\ \partial h_2/\partial x_1 & \partial h_2/\partial x_2 & \cdots & \partial h_2/\partial x_n \\ \vdots & \vdots & & \vdots \\ \partial h_n/\partial x_1 & \partial h_n/\partial x_2 & \cdots & \partial h_n/\partial x_n \end{bmatrix}$$

$$\mathbf{B}(t) \equiv \frac{\partial \mathbf{h}}{\partial \mathbf{u}} = \begin{bmatrix} \partial h_1/\partial u_1 & \partial h_1/\partial u_2 & \cdots & \partial h_1/\partial u_p \\ \partial h_2/\partial u_1 & \partial h_2/\partial u_2 & \cdots & \partial h_2/\partial u_p \\ \vdots & \vdots & & \vdots \\ \partial h_n/\partial u_1 & \partial h_n/\partial u_2 & \cdots & \partial h_n/\partial u_p \end{bmatrix}$$

Jacobians

$$\boxed{\dot{\bar{\mathbf{x}}}(t) = \mathbf{A}(t)\bar{\mathbf{x}}(t) + \mathbf{B}(t)\bar{\mathbf{u}}}$$

Linearization

- Many physical systems can be described by nonlinear differential equations
- Some nonlinear equations can be approximated by linear equations
- An example

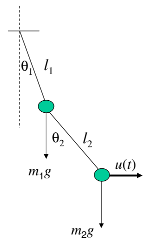
$$x_1 = \theta_1,\ x_2 = \dot{\theta}_1,\ x_3 = \theta_2,\ x_4 = \dot{\theta}_2$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l_1}\sin x_1 + \frac{m_2 g}{m_1 l_1}\cos x_3 \sin(x_3 - x_1) + \frac{1}{m_1 l_1}\sin x_3 \sin(x_3 - x_1)\cdot u$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -\frac{g}{l_2}\sin x_3 + \frac{1}{m_2 l_2}(\cos x_3)\, u$$



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Linearization

- Many physical systems can be described by nonlinear differential equations
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- An example

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l_1}\sin x_1 + \frac{m_2 g}{m_1 l_1}\cos x_3 \sin(x_3 - x_1) + \frac{1}{m_1 l_1}\sin x_3 \sin(x_3 - x_1)\cdot u$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -\frac{g}{l_2}\sin x_3 + \frac{1}{m_2 l_2}(\cos x_3)\, u$$

$$\Downarrow$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(m_1+m_2)g/(m_1 l_1) & 0 & m_2 g/m_1 l_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -g/l_2 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 l_2 \end{bmatrix} u$$

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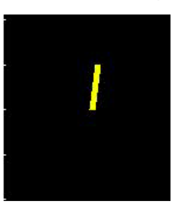
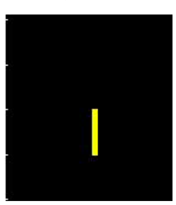
Linearization

- Many physical systems can be described by nonlinear differential equations
- Some nonlinear equations can be approximated by linear equations
- An example
- In some cases, the linearized model is a very poor approximation to the physical system





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Review of linear algebra (II)

- Matrices
  - A $q\times p$ matrix: $\hat{\mathbf{G}}(s)=\begin{bmatrix}\hat{g}_{11}(s) & \hat{g}_{12}(s) & \cdots & \hat{g}_{1p}(s)\\ \hat{g}_{21}(s) & \hat{g}_{22}(s) & \cdots & \hat{g}_{2p}(s)\\ \vdots & \vdots & & \vdots\\ \hat{g}_{q1}(s) & \hat{g}_{q2}(s) & \cdots & \hat{g}_{qp}(s)\end{bmatrix}$

    q: the number of rows
    p: the number of columns

  - Column vector: $q\times 1$ matrix
  - Row vector: $1\times p$ matrix
  - Addition and subtraction are performed element by element, while multiplication is not

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Review of linear algebra (II)

- Matrices
- Matrix multiplication

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}u & x\\ v & y\end{bmatrix}=\begin{bmatrix}au+bv & ax+by\\ cu+dv & cx+dy\end{bmatrix}$$

Product of a $1\times n$ matrix an $n\times 1$ matrix is a scalar.
Product of an $n\times 1$ matrix and a $1\times n$ matrix is an $n\times n$ matrix.

$$\begin{bmatrix}a & b & c\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=ax+by+cz \qquad \begin{bmatrix}x\\ y\\ z\end{bmatrix}\begin{bmatrix}a & b & c\end{bmatrix}=\begin{bmatrix}xa & xb & xc\\ ya & yb & yc\\ za & zb & zc\end{bmatrix}$$

Generally, $\mathbf{AB}\neq\mathbf{BA}$.

For two matrices $\mathbf{A}$ and $\mathbf{B}$ to be multiplied, they have to be compatible: The number of columns of $\mathbf{A}$ must equal to the number of rows of $\mathbf{B}$, e.g., $\mathbf{A}$: $k\times n$, $\mathbf{B}$: $n\times m$.

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Review of linear algebra (II)

- Matrices
- Matrix multiplication

$$A=\begin{bmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ 9 & 10 & 11 & 12\end{bmatrix}=\begin{bmatrix}A_1\\ A_2\\ A_3\end{bmatrix},\quad B=\begin{bmatrix}a & e\\ b & f\\ c & g\\ d & h\end{bmatrix}$$

$$B=[B_1 \quad B_2]$$

$$AB=A\begin{bmatrix}B_1 & B_2\end{bmatrix}=\begin{bmatrix}AB_1 & AB_2\end{bmatrix}$$

$$AB=\begin{bmatrix}A_1\\ A_2\\ A_3\end{bmatrix}B=\begin{bmatrix}A_1B\\ A_2B\\ A_3B\end{bmatrix}=\begin{bmatrix}A_1\begin{bmatrix}B_1 & B_2\end{bmatrix}\\ A_2\begin{bmatrix}B_1 & B_2\end{bmatrix}\\ A_3\begin{bmatrix}B_1 & B_2\end{bmatrix}\end{bmatrix}$$

$$=\begin{bmatrix}A_1B_1 & A_1B_2\\ A_2B_1 & A_2B_2\\ A_3B_1 & A_3B_2\end{bmatrix}=\begin{bmatrix}a+2b+3c+4d & e+2f+3g+4h\\ 5a+6b+7c+8d & 5e+6f+7g+8h\\ 9a+10b+11c+12d & 9e+10f+11g+12h\end{bmatrix}$$

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Review of linear algebra (II)

- Matrices
- Matrix multiplication
- Determinant
  - Determinant is a scalar defined for a square matrix

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + dhc + gbf - gec - ahf - dbi$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow \begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \\ a & b & c \\ d & e & f \end{array}$$

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Review of linear algebra (II)

- Matrices
- Matrix multiplication
- Determinant
  - Determinant is a scalar defined for a square matrix

If two rows (columns) are switched, the determinant changes the sign.

If a whole row (column) is scaled by a number k, the determinant is scaled by a number k.

If an entire row or an entire column is 0, the determinant is 0.

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Review of linear algebra (II)

- Matrices
- Matrix multiplication
- Determinant
  - Determinant is a scalar defined for a square matrix
  - Determinant of a triangular matrix

$$\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix} = 1\times5\times8\times10 = 400$$

If all elements below the diagonal (or above the diagonal) are zero, the determinant is the product of the diagonal elements.

The determinant can be simplified by making the matrix a triangular one through elementary operations that preserve the determinant.

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Review of linear algebra (II)

- Matrices
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  - Elementary operations that preserve the determinant

Add one row scaled by a number to another row

$$A=\begin{bmatrix}1&2&3\\3&2&1\\4&5&6\end{bmatrix}\xrightarrow{\text{Add row 1 to row 2}}\begin{bmatrix}1&2&3\\4&4&4\\4&5&6\end{bmatrix}$$

Add one column scaled by a number to another column

$$\begin{bmatrix}1&2&3\\4&4&4\\4&5&6\end{bmatrix}\xrightarrow{\text{Add column 2}\times(-1)\text{ to column 3}}\begin{bmatrix}1&2&1\\4&4&0\\4&5&1\end{bmatrix}$$

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Review of linear algebra (II)

- Matrices
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  - Elementary operations that preserve the determinant

Question: Why do elementary operations preserve the determinant?

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Review of linear algebra (II)

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An elementary operation is equivalent to multiplying the matrix with another one whose determinant is 1.

$$\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\x&0&1\end{bmatrix}=\begin{bmatrix}a+cx&b&c\\d+fx&e&f\\g+ix&h&i\end{bmatrix}$$

$$\det(AB)=\det A\times\det B$$
$$\det(ABC)=\det A\times\det B\times\det C$$

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