ECE 2646: Linear System Theory (3 Credits, Fall 2009)

Lecture 2: Mathematical Description of Systems

September 9, 2009

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Outline of this lecture

- A list of references for control theory
- Homework 1
- Review of last lecture
- Categories of systems
- Linear systems
- Linear time-invariant systems

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A list of references

- K. J. Astrom and R. M. Murray. Feedback Systems: An Introduction for Scientists and Engineers. Manuscript, 2007. Available online: `http://www.cds.caltech.edu/~murray/books/AM05/wiki/index.php?title=Main_Page`
- D. S. Bernstein. A Student's Guide to Classical Control. Available online: `http://www.engr.pitt.edu/electrical/faculty-staff/mao/2646/Lectures/classic_control.pdf`
- M. Dahleh, M. A. Dahleh, and G. Verghese. Lecture Notes for 6.241 Dynamic Systems and Control. Massachusetts Institute of Technology, 2003. Available online: `http://ocw.mit.edu/OcwWeb/Electrical-Engineering-and-Computer-Science/6-241Fall2003/LectureNotes/index.htm`
- G. F. Franklin, J. D. Powell, and A. Emami-Naeni. Feedback Control of Dynamic Systems, Addison-Wesley, 2002.
- B. C. Kuo. Automatic Control Systems. Prentice-Hall, 1995.
- N. Nise. Control Systems Engineering, 4th Edition, John Wiley and Sons, 2005.
- C. L. Phillips and R. D. Harbor. Feedback Control Systems, 4th Edition, Prentice Hall, 2000.

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Homework 1

- Problems 2.6, 2.10, 2.11, 2.12, and 2.17
- Due Wednesday 9/23 in class (two weeks later)

4

Review of last lecture

- Important concepts from classical control
  - Control system

A band called Control Theory

"Control Theory: The study of how to manipulate the parameters affecting the behavior of a system to produce the desired outcome."

"The music will make you feel something, strike a chord inside, and also make you want to move."

Cited from http://www.control-theory.com/bio.html

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Review of last lecture

- Important concepts from classical control
  - Control system
  - Feedback (advantages and disadvantages?)
  - Dynamical system

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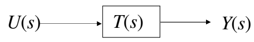
Review of last lecture

- Important concepts from classical control
  - Control system
  - Feedback
  - Dynamical system
  - Transfer function
    - Definition



$$T(s) = \frac{Y(s)}{U(s)}$$

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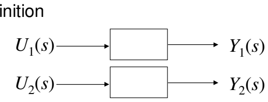
Review of last lecture

- Important concepts from classical control
  - Control system
  - Feedback
  - Dynamical system
  - Transfer function
    - Definition



Question: For any system, is it always true that

$$\frac{Y_1(s)}{U_1(s)} = \frac{Y_2(s)}{U_2(s)}?$$

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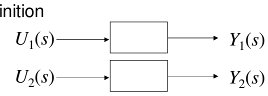
Review of last lecture

- Important concepts from classical control
  - Control system
  - Feedback
  - Dynamical system
  - Transfer function
    - Definition



Question: For an LTI (linear time-invariant) system, is it always true that

$$\frac{Y_1(s)}{U_1(s)} = \frac{Y_2(s)}{U_2(s)}?$$ Can you prove your conclusion?

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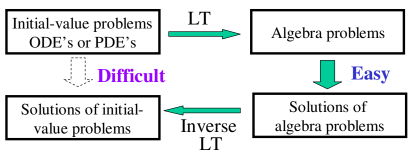
Review of last lecture

- Important concepts from classical control
  - Control system
  - Feedback
  - Dynamical system
  - Transfer function
    - Definition
    - Laplace transform and inverse Laplace transform

$$\hat{f}(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}dt \qquad f(t) = L^{-1}[\hat{f}(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}\hat{f}(s)e^{st}ds$$



Review of last lecture

- Important concepts from classical control
  - Control system
  - Feedback
  - Dynamical system
  - Transfer function
    - Definition
    - Laplace transform and inverse Laplace transform

$$\hat{f}(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}dt \qquad f(t) = L^{-1}[\hat{f}(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}\hat{f}(s)e^{st}ds$$

We seldom use the equation on the right to calculate an inverse Laplace transform; instead we use the equation of Laplace transform to construct a table of transforms for useful time functions. Then we use the table to find the inverse transform.

Review of last lecture

- Important concepts from classical control
  - Control system
  - Feedback
  - Dynamical system
  - Transfer function
    - Definition
    - Laplace transform and inverse Laplace transform

Time domain	Frequency domain
$\delta(t)$	?
$u(t)$ or $1(t)$	?
e^{-at}	?
$e^{-at}\sin(\omega t)$	?

Review of last lecture

- Important concepts from classical control
  - Control system
  - Feedback
  - Dynamical system
  - Transfer function
    - Definition
    - Laplace transform and inverse Laplace transform

Time domain	Frequency domain
$\delta(t)$	1
$u(t)$ or $1(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$

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Review of last lecture

- Important concepts from classical control
  - Control system
  - Feedback
  - Dynamical system
  - Transfer function
    - Definition
    - Laplace transform and inverse Laplace transform
    - Partial fraction expansion of a rational function (please refer to the note for Lecture 2 of ECE 1673: http://www.engr.pitt.edu/electrical/faculty-staff/mao/1673/lectures/lecture2.pdf)

Example:
$$F(s)=\frac{N(s)}{\prod_{i=1}^{n}(s-p_i)}=\frac{k_1}{s-p_1}+\cdots+\frac{k_n}{s-p_n},$$
$$\text{where } k_j=(s-p_j)F(s)\big|_{s=p_j}.$$

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Review of last lecture

- Important concepts from classical control
  - Control system
  - Feedback
  - Dynamical system
  - Transfer function
    - Definition
    - Laplace transform and inverse Laplace transform
    - Partial fraction expansion of a rational function
    - Useful theorems of Laplace transform

Differential theorem
$$L\left[\frac{df}{dt}\right]=s\hat{f}(s)-f(0^-),$$
$$\text{where } f(0^-)=\lim_{t\to 0}f(t),\quad t<0$$

Integral theorem
$$L\left[\int_0^t f(\tau)d\tau\right]=\frac{\hat{f}(s)}{s}$$

Question: What are the transfer functions of differentiator and integrator, respectively?

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Review of last lecture

- Important concepts from classical control
  - Control system
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  - Dynamical system
  - Transfer function
    - Definition
    - Laplace transform and inverse Laplace transform
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    - Useful theorems of Laplace transform

Shift theorem

$$L[f(t-t_0)u(t-t_0)] = e^{-t_0 s}\hat{f}(s)$$

Frequency shift theorem

$$L[e^{-at}f(t)] = \hat{f}(s+a)$$

Question: What is the transfer function of a pure time delay unit?

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Review of last lecture

- Important concepts from classical control
  - Control system
  - Feedback
  - Dynamical system
  - Transfer function
    - Definition
    - Laplace transform and inverse Laplace transform
    - Partial fraction expansion of a rational function
    - Useful theorems of Laplace transform

Theorem of convolution integral

$$L^{-1}[\hat{f}_1(s)\hat{f}_2(s)] = \int_0^t f_1(t-\tau)f_2(\tau)d\tau = \int_0^t f_1(\tau)f_2(t-\tau)d\tau$$

Convolution product in the time domain ⇔ Product in the frequency domain

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Review of last lecture

- Important concepts from classical control
  - Control system
  - Feedback
  - Dynamical system
  - Transfer function
  - Tome-domain analysis (e.g. root-locus plot)
  - Frequency-domain analysis (e.g. Bode plot and Nyquist diagram)

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Review of last lecture

- Important concepts from classical control
  - Control system
  - Feedback
  - Dynamical system
  - Transfer function
  - Tome-domain analysis
  - Frequency-domain analysis
  - Stability
    - Bounded-input, bounded-output (BIBO) stability

> Criterion for BIBO stability: A linear time-invariant system is BIBO stable provided all roots of the system characteristic equation (poles of the closed-loop transfer function) lie in the left half of the s-plane [Why?]

Review of last lecture

- Important concepts from classical control
  - Control system
  - Feedback
  - Dynamical system
  - Transfer function
  - Tome-domain analysis
  - Frequency-domain analysis
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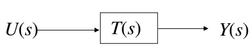

$$T(s) = \frac{P(s)}{Q(s)}, \quad Q(s) = a_n \prod_{i=1}^{n} (s - p_i)$$

$$Y(s) = T(s)U(s) = \frac{P(s)}{a_n \prod_{i=1}^{n} (s - p_i)} U(s) = \frac{k_1}{s - p_1} + \cdots + \frac{k_n}{s - p_n} + Y_r(s)$$

Characteristic equation: $Q(s) = 0$
System roots or system poles: p_i, $i = 1,\ldots,n$
Forced response: $Y_r(s)$ [the sum of terms, in the partial-fraction expansion, that originate in the poles of $U(s)$]

Review of last lecture

- Important concepts from classical control
  - Control system
  - Feedback
  - Dynamical system
  - Transfer function
  - Tome-domain analysis
  - Frequency-domain analysis
  - Stability
  - Sensitivity
  - Commonly used controllers

Review of last lecture

- Important concepts from classical control
- What is new in this course?
  - State-space description

Question: What are the advantages of using state-space approaches?

- State-space form is a convenient way to work with complex dynamics; matrix format is easy to use on computers
- Transfer functions only deal with input/output behavior, while state-space form provides easy access to the internal features and response of the system
- State-space approach is great for MIMO (multi-input multi-output) system, which are very hard to work with using transfer functions

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Review of last lecture

- Important concepts from classical control
- What is new in this course?
  - State-space description

Question: What are the advantages of using state-space approaches?

- State variable form is a convenient way to work with complex dynamics; matrix format is easy to use on computers
- Transfer functions only deal with input/output behavior, while state-space form provides easy access to the internal features and response of the system
- State-space approach is great for MIMO (multi-input multi-output) system, which are very hard to work with using transfer functions
- State space can be used to study more general models: The ODEs do not have to be linear
- State space introduces the ideas of geometry into differential equations (remember the phase plane in physics?)

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Review of last lecture

- Important concepts from classical control
- What is new in this course?
  - State-space description

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

  - Feedback based on state variables (e.g., $\mathbf{u} = \mathbf{r} - \mathbf{K}\mathbf{x}$)
  - State-space based time-domain techniques

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Review of last lecture

- Important concepts from classical control
- What is new in this course?
- Review of linear algebra (I)
  - Exercise: Find the determinant of the following matrix

$$D = \begin{bmatrix} \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

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Categories of systems

- Single-input single-output (SISO) system, multi-input multi-output (MIMO) system, SIMO system, and MISO system
- Continuous-time system, discrete-time system, and hybrid system

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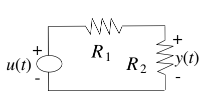
Categories of systems

- Single-input single-output (SISO) system, multi-input multi-output (MIMO) system, SIMO system, and MISO system
- Continuous-time system, discrete-time system, and hybrid system
- Memoryless system and system that has memory
  - A system is called a memoryless system if its output $\mathbf{y}(t_0)$ depends only on the input applied at t_0; it is independent of the input applied before or after t_0
    - Example: A matrix can be interpreted as a memoryless system relating an input vector to an output vector



Question: Does a purely resistive circuit (shown in the left, with $u(t)$ as input and $y(t)$ as output) have memory? What if we add a capacitor in the circuit?

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Categories of systems

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- Memoryless system and system that has memory
  - A system is called a memoryless system if its output $y(t_0)$ depends only on the input applied at t_0; it is independent of the input applied before or after t_0
  - Most systems have memory: Their current output may depend on past, current, and even future inputs
    - Question: Does the following system have memory?

$$m\frac{d^2y(t)}{dt^2}=u(t)$$

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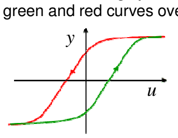
Question: Does the following system have memory?
What if the green and red curves overlap each other?



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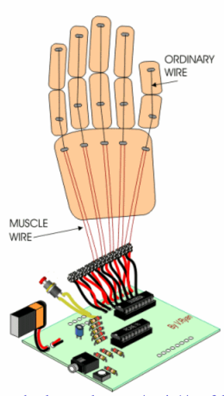
Smart material with memory



Picture from http://www.technologystudent.com/equip1/sma2.htm

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Categories of systems

- Single-input single-output (SISO) system, multi-input multi-output (MIMO) system, SIMO system, and MISO system
- Continuous-time system, discrete-time system, and hybrid system
- Memoryless system and system that has memory
- Causal system and noncausal system
  - A system is called a causal or nonanticipatory system if its current output depends on past and current but not future input
  - If a system is noncausal, then its current output will depend on future input (no physical system has such capability!)

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Categories of systems

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Question: Consider a discrete-time system that does the operation of smoothing its input signals by averaging as follows:

$$y[t] = \frac{x[t-1] + x[t] + x[t+1]}{3}$$

Is this system causal?

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Categories of systems

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- Continuous-time system, discrete-time system, and hybrid system
- Memoryless system and system that has memory
- Causal system and noncausal system
- Concept of state
  - Generally, current output of a causal system $\mathbf{y}(t)$ is affected by input from $-\infty$ to time t (very inconvenient to track $\mathbf{u}(t)$ from $-\infty$ to t)

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Categories of systems

- Single-input single-output (SISO) system, multi-input multi-output (MIMO) system, SIMO system, and MISO system
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- Concept of state
  - Generally, current output of a causal system $\mathbf{y}(t)$ is affected by input from -∞ to time t (very inconvenient to track $\mathbf{u}(t)$ from -∞ to t)
  - The state $\mathbf{x}(t_0)$ of a system at time t_0 is the information at t_0 that, together with the input $\mathbf{u}(t)$, for $t \geq t_0$, determines uniquely the output $\mathbf{y}(t)$ for all $t \geq t_0$

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  - The state summaries the effect of past input on future output: Using the state at t_0, we can express the input and output of a system as

$$\left.\begin{array}{l}\mathbf{x}(t_0)\\ \mathbf{u}(t),\ t \geq t_0\end{array}\right\} \rightarrow \mathbf{y}(t),\ t \geq t_0$$

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- Continuous-time system, discrete-time system, and hybrid system
- Memoryless system and system that has memory
- Causal system and noncausal system
- Concept of state
- Lumped system and distributed system
  - A system is said to be lumped if its number of state variables is a finite vector
    - Example:

$$m\frac{d^2 y(t)}{dt^2} = u(t)$$

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Categories of systems

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- Concept of state
- Lumped system and distributed system
  - A system is said to be lumped if its number of state variables is a finite vector
  - A system is called a distributed system if its state has infinitely many state variables
    - Example:

$$y(t) = u(t-1)$$

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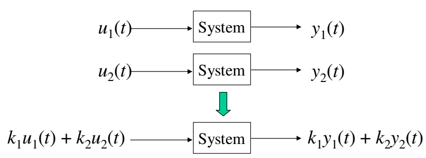
Linear systems

- Definition
  - A system is linear if superposition applies



Question: Is $y(t) = u(t) + 2$ a linear system?

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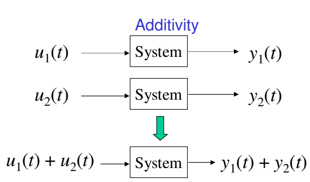
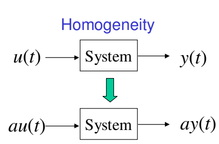
Linear systems

- Definition
  - Equivalently, a system is linear if the additivity and homogeneity properties hold





Question: Can additivity be derived from homogeneity? Can homogeneity be derived from additivity?

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Linear systems

- Definition
  - A system is called a linear system if for every t_0 and any two state-input–output pairs

$$\left.\begin{matrix}\mathbf{x}_i(t_0) \\ \mathbf{u}_i(t),\ \ t \ge t_0\end{matrix}\right\} \rightarrow \mathbf{y}_i(t),\ \ t \ge t_0$$

for $i = 1, 2$, we have

$$\left.\begin{matrix}\alpha_1\mathbf{x}_1(t_0)+\alpha_2\mathbf{x}_2(t_0) \\ \alpha_1\mathbf{u}_1(t)+\alpha_2\mathbf{u}_2(t),\ \ t \ge t_0\end{matrix}\right\} \rightarrow \alpha_1\mathbf{y}_1(t)+\alpha_2\mathbf{y}_2(t),\ \ t \ge t_0$$

for any real constants α_1 and α_2

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Linear systems

- Definition
- Zero-input response and zero-state response
  - Zero-input response

$$\left.\begin{matrix}\mathbf{x}(t_0) \\ \mathbf{u}(t)=0,\ \ t \ge t_0\end{matrix}\right\} \rightarrow \mathbf{y}_{zi}(t),\ \ t \ge t_0$$

  - Zero-state response

$$\left.\begin{matrix}\mathbf{x}(t_0)=0 \\ \mathbf{u}(t),\ \ t \ge t_0\end{matrix}\right\} \rightarrow \mathbf{y}_{zs}(t),\ \ t \ge t_0$$

Response = zero-input response + zero-state response

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Linear systems

- Definition
- Zero-input response and zero-state response
- Input-output description

For a linear SISO system:

$$y(t)=\int_{-\infty}^{+\infty} g(t,\tau)u(\tau)d\tau$$

In the above equation, $g(t,\tau)$ is the response excited by an impulse and is called the impulse response. In $g(t,\tau)$, the first variable denotes the time at which the output is observed, and the second variable denotes the time at which the impulse input is applied.

Question: Can you prove the above equation? (Hint: page 9 of the text book)

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Linear systems

- Definition
- Zero-input response and zero-state response
- Input-output description

For a linear SISO system: $y(t)=\int_{-\infty}^{+\infty} g(t,\tau)u(\tau)d\tau$

Question: If the system is causal, what is the value of $g(t,\tau)$ for $t<\tau$?

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Linear systems

- Definition
- Zero-input response and zero-state response
- Input-output description

For a linear, causal, SISO system: $y(t)=\int_{-\infty}^{t} g(t,\tau)u(\tau)d\tau$

Causal $\Leftrightarrow$ $g(t,\tau)=0$ for $t<\tau$

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Linear systems

- Definition
- Zero-input response and zero-state response
- Input-output description

For a linear, causal, SISO system relaxed at t_0: $y(t)=\int_{t_0}^{t} g(t,\tau)u(\tau)d\tau$

A system is said to be relaxed at t_0 if its initial state is 0 at t_0

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Linear systems

- Definition
- Zero-input response and zero-state response
- Input-output description
- Impulse response matrix

For a linear, causal, MIMO system (with p input and q output) relaxed at t_0:

$$\mathbf{y}(t) = \int_{t_0}^{t} \mathbf{G}(t,\tau)\mathbf{u}(\tau)d\tau$$

$$\mathbf{G}(t,\tau) = \begin{bmatrix} g_{11}(t,\tau) & g_{12}(t,\tau) & \cdots & g_{1p}(t,\tau) \\ g_{21}(t,\tau) & g_{22}(t,\tau) & \cdots & g_{2p}(t,\tau) \\ \vdots & \vdots & & \vdots \\ g_{q1}(t,\tau) & g_{q2}(t,\tau) & \cdots & g_{qp}(t,\tau) \end{bmatrix}$$

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Linear time-invariant systems

- Definition
  - A system is said to be time invariant if for every state-input-output pair

$$\left.\begin{matrix} \mathbf{x}(t_0) \\ \mathbf{u}(t),\ t \geq t_0 \end{matrix}\right\} \rightarrow \mathbf{y}(t),\ t \geq t_0$$

and any T, we have

$$\left.\begin{matrix} \mathbf{x}(t_0+T) \\ \mathbf{u}(t-T),\ t \geq t_0+T \end{matrix}\right\} \rightarrow \mathbf{y}(t-T),\ t \geq t_0+T$$

  - The above definition means that if the initial state is shifted to time t_0+T and the same input waveform is applied from t_0+T, the output waveform will be the same except that it starts to appear from time t_0+T

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Linear time-invariant systems

- Definition
  - A system is said to be time invariant if for every state-input-output pair

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  - The above definition means that if the initial state is shifted to time t_0+T and the same input waveform is applied from t_0+T, the output waveform will be the same except that it starts to appear from time t_0+T
  - Examples of time-variant systems: burning rocket and brain

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Linear time-invariant systems

- Definition
- Input-output description

For a causal, SISO, LTI system relaxed at 0:

$$g(t,\tau)=g(t+T,\tau+T)=g(t-\tau,0)\equiv g(t-\tau)$$

$$y(t)=\int_0^t g(t,\tau)u(\tau)d\tau \rightarrow y(t)=\int_0^t g(t-\tau)u(\tau)d\tau$$

Convolution integral

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Linear time-invariant systems

- Definition
- Input-output description

Question: If an LTI system is causal, what is the value of $g(t)$ for $t<0$?

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Linear time-invariant systems

- Definition
- Input-output description

The condition for an LTI system to be causal is $g(t)=0$ for $t<0$

Question: The impulse response of an ideal lowpass filter is given by

$$g(t)=2\omega\frac{\sin 2\omega(t-t_0)}{2\omega(t-t_0)}.$$

Is the ideal lowpass filter causal?

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Linear time-invariant systems

- Definition
- Input-output description
- Transfer-function matrix

For a causal, MIMO, LTI system (with p input and q output) relaxed at 0:

$$\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)$$

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \hat{g}_{11}(s) & \hat{g}_{12}(s) & \cdots & \hat{g}_{1p}(s) \\ \hat{g}_{21}(s) & \hat{g}_{22}(s) & \cdots & \hat{g}_{2p}(s) \\ \vdots & \vdots & & \vdots \\ \hat{g}_{q1}(s) & \hat{g}_{q2}(s) & \cdots & \hat{g}_{qp}(s) \end{bmatrix}$$

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