ECE 1673: Linear Control Systems (4 Credits, Spring 2009)

Lecture 9: Root Locus Rules

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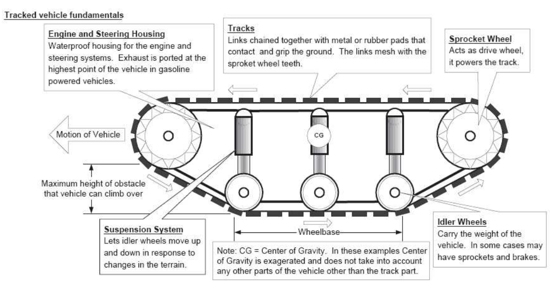
Questions from the homework

- Time constant and frequency response
- Time constant of second order system (with complex poles and with real poles)

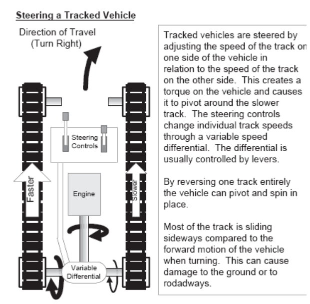
About Lab 2

- What is tracked vehicle?



About Lab 2

- What is tracked vehicle?
- Turning control of a tracked vehicle
  - Internal view: nonlinear and complex relationships
  - External view: simplified LTI input-output relationships

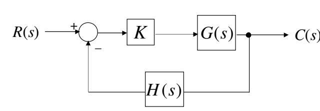


About last lecture

- Stability by simulation
- Introduction to root locus
  - Definition: A root locus of a system is a plot of the roots of the system characteristic equation (the poles of the closed-loop transfer function) as some parameter of the system is varied

About last lecture

- Stability by simulation
- Introduction to root locus
  - Definition
  - General system for root locus



$$1 + KG(s)H(s) = 0$$

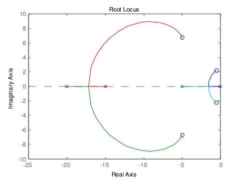
Open-loop function

Characteristic equation

About last lecture

- Stability by simulation
- Introduction to root locus
  - Definition
  - General system for root locus
  - Matlab program for root-locus plot: ??
  - Observations: ??

$$G(s)=\frac{(s^2+s+5)(s^2+10s+70)}{s(s+5)(s+15)(s+20)},$$
$$H(s)=1$$



- Root-locus example: pitch control of the 1903 Wright Flyer

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Outline of this lecture

- Some important concepts
- Six rules for root-locus plot

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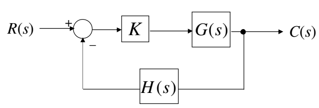
Some important concepts

- Angle criterion



$$\sum(\text{all angles from the finite zeros})-\sum(\text{all angles from the finite poles}) = r(180^\circ) \quad r=\pm 1,\pm 3,\ldots$$
where the poles and zeros are those of the open-loop function

$$\angle KG(s)H(s)=r(180^\circ), \text{ where } r=\pm 1,\pm 3,\ldots$$

Why?

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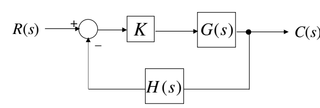
Some important concepts

- Angle criterion



$$\angle KG(s)H(s)=r(180^\circ), \text{ where } r=\pm 1,\pm 3,\ldots$$

Why?

$$1+KG(s)H(s)=0$$
$$KG(s)H(s)=-1$$

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Some important concepts

- Angle criterion
- Magnitude criterion

$$|KG(s)H(s)|=1$$

Why?

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Some important concepts

- Angle criterion
- Magnitude criterion
- Axis crossings
  - The points where the root locus intersects the imaginary axis indicate the values of K at which the closed loop system is marginally stable

Question: How to find the imaginary-axis crossing points?

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Some important concepts

- Angle criterion
- Magnitude criterion
- Axis crossings
  - The points where the root locus intersects the imaginary axis indicate the values of K at which the closed loop system is marginally stable

Question: How to find the imaginary-axis crossing points?

$$1 + KG(j\omega)H(j\omega) = 0$$

Rule 1

- The root locus is symmetrical with respect to the real axis

Why?

Rule 1

- The root locus is symmetrical with respect to the real axis

Why?

Properties of complex conjugate :

$$\overline{uv} = (\bar{u})(\bar{v})$$

$$\overline{u+v} = \bar{u} + \bar{v}$$

$$\overline{\left(\frac{u}{v}\right)} = \bar{u}/\bar{v}$$

Rule 2

- The root locus originates on the poles of $G(s)H(s)$ (for $K = 0$) and terminates on the zeros of $G(s)H(s)$ (as $K \to \infty$), including those zeros at infinity

Why?

Rule 2

- The root locus originates on the poles of $G(s)H(s)$ (for $K = 0$) and terminates on the zeros of $G(s)H(s)$ (as $K \to \infty$), including those zeros at infinity

Why?

$$1 + KG(s)H(s) = 1 + \frac{Kb_m(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)} = 0$$

$$(s - p_1)\cdots(s - p_n) + Kb_m(s - z_1)\cdots(s - z_m) = 0$$

Rule 3

- If the open-loop function has zeros at infinity, the root locus approaches asymptotes as K approaches infinity. The asymptotes are located at the angles

$$\theta = \frac{r180^\circ}{\alpha}, \quad \alpha = n - m, \quad r = \pm 1, \pm 3, \ldots$$

and these asymptotes intersect the real axis at the point

$$\sigma_a = \frac{\text{(sum of finite poles) - (sum of finite zeros)}}{\text{(number of finite poles) - (number of finite zeros)}}$$

Why? Hint: Angle criterion

Rule 3

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Exercise: Angles of asymptotes

α	Angles of asymptotes	Examples $[G(s)H(s)]$
0	No asymptotes	
1	??	$1/(s+1)$
2	??	$1/[s(s+1)]$
3	??	$1/s^3$
4	??	$1/[s(s+1)(s+2)(s+3)]$

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Rule 4

- The root locus includes all points on the real axis to the left of an odd number of real critical frequencies (poles and zeros)

Why?

Hint: Angle criterion

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Rule 5

- The breakaway points on a root locus will appear among the roots of the polynomial obtained from either

$$\frac{d[G(s)H(s)]}{ds} = 0$$

Why?

or, equivalently,

$$N(s)D'(s) - N'(s)D(s) = 0,$$

where $N(s)$ and $D(s)$ are the numerator and denominator polynomials, respectively, of $G(s)H(s)$

Points at which two or more branches of the locus come together and then part (break away)

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Rule 5

- The breakaway points on a root locus will appear among the roots of the polynomial obtained from either $\frac{d[G(s)H(s)]}{ds} = 0$

  or, equivalently, $N(s)D'(s) - N'(s)D(s) = 0,$

  where $N(s)$ and $D(s)$ are the numerator and denominator polynomials, respectively, of $G(s)H(s)$
- More on breakaway points
  - Break points often occur on the real axis, but they may appear anywhere in the s-plane
  - The loci that approach/diverge from a breakaway point do so at angles spaced equally about the break point
  - The angles at which they arrive/leave are a function of the number of loci that approach/diverge from the breakaway point

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Rule 6

- Loci will depart from a pole p_j (arrive at a zero z_j) of $G(s)H(s)$ at the angle θ_d (θ_a), where

$$\theta_d = \sum_i \theta_{zi} - \sum_{i, i \neq j} \theta_{pi} + r(180^\circ)$$

$$\theta_a = \sum_i \theta_{pi} - \sum_{i, i \neq j} \theta_{zi} + r(180^\circ)$$

and where $r = \pm 1, \pm 3, \ldots$ and θ_{pi} (θ_{zi}) represent the angles from pole p_i (zero z_i), respectively, to p_j (zero z_j)

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Disturbance, robustness, and walking robot

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References

- C. L. Phillips and R. D. Harbor. Feedback Control Systems, 4th Edition, Prentice Hall, 2000.
- http://www.bostondynamics.com/content/sec.php?section=BigDog
- http://www.cpdee.ufmg.br/~palhares/aula6_csl.pdf