ECE 1673: Linear Control Systems (4 Credits, Spring 2009)

Lecture 6: Disturbances and Sensitivity

January 26, 2009

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Outline of this lecture

- Questions from Homework 1
- Review of last lecture
- Routh criterion for stability
- Structure of closed-loop control systems (revisit and more)
- Sensitivity
- Disturbance rejection

Questions from Homework 1

Question: What is the Laplace transform of $df(t)/dt$ given

(1) $f(t) = 4e^{-2(t-3)}$,

(2) $f(t) = 4e^{-2(t-3)}u(t)$,

(3) $f(t) = 4e^{-2(t-3)}u(t-3)$?

Questions from Homework 1

Question: What are the inverse Laplace transforms of

$$F_1(s) = \frac{2s+1}{s^2+2s+10},$$

$$F_2(s) = \frac{s-30}{s(s^2+4s+29)}?$$

Questions from Homework 1

Remarks on Problems 2.12 and 2.13:

(1) Transfer functions in a system with multiple inputs or multiple outputs (rule of superposition);

(2) A transfer function should be independent of its input or output.

Review last lecture

- What can we learn from a transfer function?

Review of last lecture

- What can we learn from a transfer function?
  - Model of the system dynamics in differential equations
  - Impulse response
  - Steady-state response to a step input
  - Frequency response
  - Characteristics of the natural response (transient response)
  - Stability of the system

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Review last lecture

- What can we learn from a transfer function?
- BIBO (bounded-input, bounded-output) stability
  - Definition
    - A system is BIBO stable, if, for every bounded input, the output remains bounded for all time
    - The concept of BIBO stability is not appropriate to describe stability in all nonlinear systems

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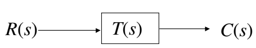
Review last lecture

- What can we learn from a transfer function?
- BIBO stability
  - Definition
  - Criteria



$$T(s)=\frac{P(s)}{Q(s)},\quad Q(s)=a_n\prod_{i=1}^{n}(s-p_i)$$

$$C(s)=T(s)R(s)=\frac{P(s)}{a_n\prod_{i=1}^{n}(s-p_i)}R(s)=\frac{k_1}{s-p_1}+\cdots+\frac{k_n}{s-p_n}+C_r(s)$$

Some terms
- Characteristic equation: $Q(s)=0$
- System roots or system poles: $p_i,\ i=1,\ldots,n$
- Forced response: $C_r(s)$ (the sum of terms, in the partial-fraction expansion, that originate in the poles of $R(s)$)

Review last lecture

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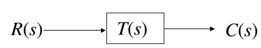

$$T(s)=\frac{P(s)}{Q(s)},\quad Q(s)=a_n\prod_{i=1}^{n}(s-p_i)$$

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A linear time-invariant system is BIBO stable provided all roots of the system characteristic equation (poles of the closed-loop transfer function) lie in the left half of the s-plane (why?)

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Review last lecture

- What can we learn from a transfer function?
- BIBO stability
  - Definition
  - Criteria
  - Examples:

$$\frac{s-2}{(s+2)(s^2+4s+13)},\ \frac{1}{s^2-3},\ \frac{1}{s},\ \frac{1}{s^2+3}$$

$$\frac{e^{-2s}}{s^2+4s+13}$$

  - Marginally stable

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Routh stability criterion

- The Routh criterion is a method for determining stability for systems with an nth-order characteristic equation of the form:

$$a_n s^n + a_{n-1}s^{n-1}+\cdots+a_1 s+a_0=0$$

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Routh stability criterion

- The Routh criterion is a method for determining stability for systems with an nth-order characteristic equation of the form: $a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$
- Routh table

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$
s^{n-3}	c_1	c_2	c_3	$\cdots$
.	.	.	.	$\cdots$

$$b_1 = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix},\quad b_2 = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix},\ \cdots$$

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix},\quad c_2 = -\frac{1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix},\ \cdots$$

  - The determinant in the expression for the ith coefficient in a row is formed from the first column and the $(i+1)$th column of the two preceding rows
  - The table is continued horizontally and vertically until only zeros are obtained

Routh stability criterion

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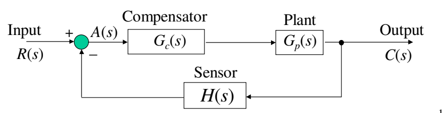
- Routh criterion
  - All the roots of the characteristic equation have negative real parts if and only if the elements of the first column of the Routh table have the same sign. Otherwise, the number of roots with positive real parts is equal to the number of changes of sign
  - Exercise:

$$Q(s) = (s+2)(s^2 - s + 4) = s^3 + s^2 + 2s + 8 = 0$$

Structure of closed-loop control systems

- Components of a closed-loop control system
  - Plant $G_p(s)$
  - Sensor $H(s)$ (usually responding much faster than plant; modeled often as a pure gain H_k)
  - Compensator or controller $G_c(s)$
  - System input $R(s)$
  - System output $C(s)$
  - Actuating signal $A(s)$

Question: What is the transfer function of this closed-loop system?



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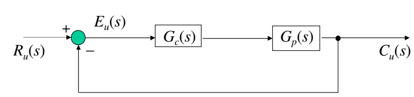
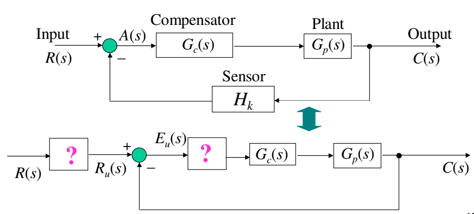
  - System output $C(s)$
  - Actuating signal $A(s)$
  - Desired value of system output $C_d(s)$
  - System error $E(s) = C_d(s) - C(s)$
- Unity feedback system



Structure of closed-loop control systems

- Components of a closed-loop control system
- Unity feedback system
  - Unity feedback model (equivalent model having unity feedback)



Structure of closed-loop control systems

- Components of a closed-loop control system
- Unity feedback system
  - Unity feedback model (equivalent model having unity feedback)
  - Under what assumptions can we use the unity feedback model?
    - For the application considered, the sensor can be modeled as a pure gain H_k (why?)
    - The equivalent forward path transfer function within the loop is $H_k G_c(s) G_p(s)$
    - The input of the unity feedback model, $R_u(s)$, is the physical system input $R(s)$ multiplied by $1/H_k$

Sensitivity

- Definition
  - The general topic of system characteristics changing with system parameter variations
  - The sensitivity function of a characteristic W with respect to the parameter b

$$S_b^W = \frac{\partial W}{\partial b}\frac{b}{W}$$

  - Interpretation of the above equation: the ratio of the percent change in the characteristic W to the percent change in the parameter b

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$$S_b^W = \frac{\partial W}{\partial b}\frac{b}{W}$$

  - Interpretation of the above equation: the ratio of the percent change in the characteristic W to the percent change in the parameter b
  - Exercise: Given $T(s) = \dfrac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)H(s)}$,

  What is $S_{G_p}^T$?

Sensitivity

- Definition
- Sensitivity of the system (closed-loop) transfer function $T(s)$ to the plant transfer function $G_p(s)$

$$S_{G_p}^T(j\omega) = \frac{1}{1+G_c(j\omega)G_p(j\omega)H(j\omega)}$$

Question 1: How to reduce this sensitivity?

Question 2: What is the sensitivity of the open-loop transfer function $G(s) = G_c(s)G_p(s)$ to $G_p(s)$?

Sensitivity

- Definition
- Sensitivity of the system (closed-loop) transfer function $T(s)$ to the plant transfer function $G_p(s)$

$$S_{G_p}^T(j\omega) = \frac{1}{1+G_c(j\omega)G_p(j\omega)H(j\omega)}$$

  - At frequencies within the system bandwidth, we would like for the loop gain (G_cG_pH) to be as large as possible, to reduce the sensitivity of the system characteristics to the parameters within the plant
  - The sensitivity of $T(s)$ to a parameter b in $G_p(s)$:

$$S_b^T = \frac{\partial T}{\partial b}\frac{b}{T} = \frac{\partial T}{\partial G_p}\frac{\partial G_p}{\partial b}\frac{b}{T}$$

Sensitivity

- Definition
- Sensitivity of the system (closed-loop) transfer function $T(s)$ to the plant transfer function $G_p(s)$

$$S_{G_p}^T = \frac{1}{1+G_cG_pH}$$

  - At frequencies within the system bandwidth, we would like for the loop gain (G_cG_pH) to be as large as possible, to reduce the sensitivity of the system characteristics to the parameters within the plant
- Sensitivity of $T(s)$ to the sensor $H(s)$
  - For the system sensitivity with respect to the sensor to be small, the loop gain must be small

$$S_H^T = \frac{-G_cG_pH}{1+G_cG_pH}$$

Sensitivity

- Definition
- Sensitivity of the system (closed-loop) transfer function $T(s)$ to the plant transfer function $G_p(s)$

$$S_{G_p}^T = \frac{1}{1+G_cG_pH}$$

- Sensitivity of $T(s)$ to the sensor $H(s)$

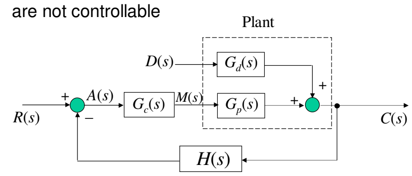
$$S_H^T = \frac{-G_cG_pH}{1+G_cG_pH}$$

- Some design considerations
  - The system cannot be insensitive to both the plant and the sensor
  - To solve the above problem, we can choose high-quality stable components for the sensor

Disturbance rejection

Disturbance rejection

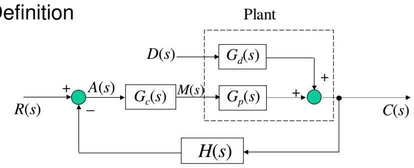
- Definition
  - Disturbances are inputs that influence the plant output but are not controllable



Question: What is the transfer function from $D(s)$ to $C(s)$?

Disturbance rejection

- Definition



$$C(s) = T(s)R(s) + T_d(s)D(s)$$

$$= \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)H(s)}R(s) + \frac{G_d(s)}{1+G_c(s)G_p(s)H(s)}D(s)$$

Disturbance rejection

- Definition
- Ways to reject disturbances
  - To increase the loop gain

$$C(s) = T(s)R(s) + T_d(s)D(s)$$

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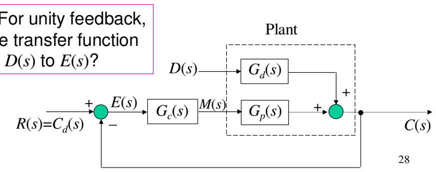
Question: For unity feedback, what is the transfer function from $D(s)$ to $E(s)$?



Disturbance rejection

- Definition
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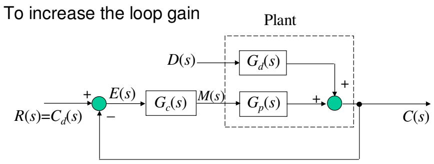
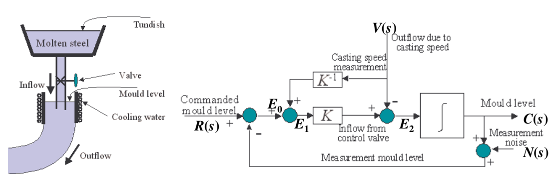


$$E(s) = \frac{1}{1+G_c(s)G_p(s)}R(s) + \frac{G_d(s)}{1+G_c(s)G_p(s)}D(s)$$

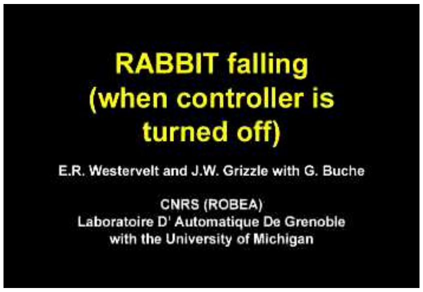
Disturbance rejection

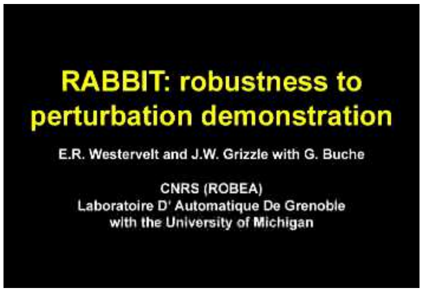
- Definition
- Ways to reject disturbances
  - To increase the loop gain
  - To use the feedforward controller



Disturbance rejection

- Definition
- Ways to reject disturbances
  - To increase the loop gain
  - To use the feedforward controller
  - To switch controllers or control strategies
    - An experiment: human strategies to keep balance





Summary: advantages of feedback (revisit)

- Feedback can change the dynamic response and make the system faster (but sometimes less stable)
- Feedback can make an unstable system stable
- Feedback can make the system less sensitive to variations in the plant parameters
- Feedback can reduce the system errors to constant disturbances

References

- C. L. Phillips and R. D. Harbor. Feedback Control Systems, 4th Edition, Prentice Hall, 2000.
- http://www.eecs.umich.edu/~grizzle/westervelt_thesis/movies/