ECE 1673: Linear Control Systems (4 Credits, Spring 2009)

Lecture 4: System Responses of First-Order and Second-Order Systems

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Homework 2

- Problems 4.5 (a)-(d) and (f), 5.4, 5.6 (a)-(c), and 5.11 in the text book
- Due 1/26 Monday

Some terms in control systems

- **Compensator:** another name of controller in some applications of control systems
- **Regulator:** a type of control systems used to maintain a physical variable at some constant value in the presence of disturbances
- **Servomechanism:** a type of control systems used to follow, or track, some desired time function
- **Actuator:** [a part of plant or not?]

Review of last lecture

- System modeling
  - Transfer functions

Questions:
(1) What type of systems can be characterized by transfer functions?
(2) How to find out the transfer function of a system?

Exercise: Find out the transfer functions of the following systems:

(1) $$I\frac{d^2y(t)}{dt^2}+B\frac{dy(t)}{dt}+Ky(t)=x(t)$$

(2) $$y(t)=x(t-T)u(t-T)$$

Review of last lecture

- System modeling
  - Transfer functions

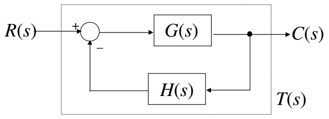
Exercise: Find out the relation between u and y:

$$I\frac{d^2y(t)}{dt^2}+B\frac{dy(t)}{dt}+Ky(t)=\frac{dx(t)}{dt}+x(t)$$

$$M\frac{d^2x(t)}{dt^2}+Dx(t)=u(t)$$

Review of last lecture

- System modeling
- Block diagrams and signal flow graphs
  - Finding system transfer functions involves solving simultaneously algebra equations



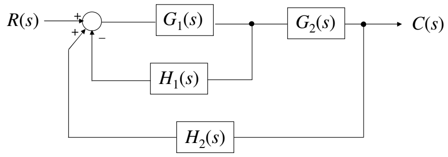
$$T(s)=\frac{\text{Gain of the feedforward path}}{1-\text{Gain of the loop}}$$

$$=\frac{G(s)}{1-(-1)G(s)H(s)}=\frac{G(s)}{1+G(s)H(s)}$$

Review of last lecture

- System modeling
- Block diagrams and signal flow graphs

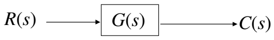
Exercise: Find out the transfer function of the following system:



Outline of this lecture

- An exercise
- General consideration on system responses
- Time response of first-order systems
- Time response of second-order systems
- Time response specifications in design
- Frequency response of systems

An exercise



- Find time responses $c(t)$ for

$$G(s)=\frac{K}{\tau s+1}$$

$$\text{(i) } r(t)=\delta(t);\quad \text{(ii) } r(t)=u(t);$$

$$\text{(iii) } r(t)=t;\quad \text{(iv) } r(t)=A\cos\omega_1 t.$$

General considerations on system responses

- Why do we emphasize first-order and second-order systems?
  - Higher-order systems can be considered to be sum of the responses of first- and second-order systems

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General considerations on system responses

- Why we emphasize first-order and second-order systems?
- Common input signals under investigation
  - Step function
  - Ramp function
  - Sinusoidal function (frequency response)

Time response of first-order systems

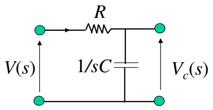
- First-order systems

dc gain

$$G(s)=\frac{C(s)}{R(s)}=\frac{b_0}{s+a_0}=\frac{K}{\tau s+1}$$

Time constant

  - An example:



$$G(s)=\frac{V_C(s)}{V(s)}=\frac{1/(Cs)}{R+1/(Cs)}=\frac{1}{RCs+1}$$

Question: What does this circuit often used for?

Time response of first-order systems

- First-order systems

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_0}{s + a_0} = \frac{K}{\tau s + 1}$$

  - An example
  - Initial conditions

$$\left(s + \frac{1}{\tau}\right) C(s) = \frac{K}{\tau} R(s) \Longrightarrow \frac{dc(t)}{dt} + \frac{1}{\tau} c(t) = \frac{K}{\tau} r(t)$$

With zero initial condition

$$\Downarrow$$

$$C(s) = \frac{c(0)}{s + (1/\tau)} + \frac{K}{\tau s + 1} R(s) \Longleftarrow sC(s) - c(0) + \frac{1}{\tau} C(s) = \frac{K}{\tau} R(s)$$

$$= \frac{K}{\tau s + 1}\left(R(s) + c(0)\frac{\tau}{K}\right) \Longrightarrow$$

With initial term

Initial condition is equivalent to an input of impulse function

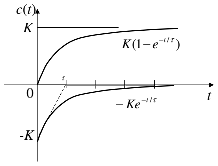
Time response of first-order systems

- First-order systems $G(s) = \frac{C(s)}{R(s)} = \frac{K}{\tau s + 1}$
- Step response

$$R(s) = 1/s,$$

$$C(s) = \frac{1}{s}\frac{K/\tau}{s + 1/\tau} = \frac{K}{s} - \frac{K}{s + 1/\tau},$$

$$c(t) = K(1 - e^{-t/\tau}), \quad t > 0$$



The limit of $c(t)$ as t goes to infinity is called the final value, or steady-state value of the response.

The parameter τ is called time constant; we may consider an exponential term to be zero after four time constants.

Time response of first-order systems

- First-order systems $G(s) = \frac{C(s)}{R(s)} = \frac{K}{\tau s + 1}$
- Step response

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$$C(s) = \frac{1}{s}\frac{K/\tau}{s + 1/\tau} = \frac{K}{s} - \frac{K}{s + 1/\tau},$$

$$c(t) = K - Ke^{-t/\tau}, \quad t > 0$$

Forced response or steady-state response

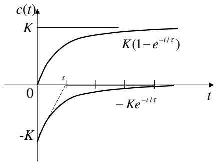
Natural response or transient response

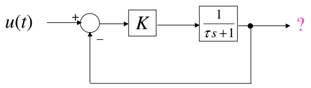
Time response of first-order systems

- First-order systems $G(s) = \frac{C(s)}{R(s)} = \frac{K}{\tau s + 1}$
- Step response

$$c(t) = K(1 - e^{-t/\tau}), \quad t > 0$$

  - An example: realizing fast step response with a simple feedback controller





Time response of first-order systems

- First-order systems $G(s) = \frac{C(s)}{R(s)} = \frac{K}{\tau s + 1}$
- Step response
- System dc gain
  - The system dc gain is the steady-state gain to a constant input for the case the output has a final value, and it is equal to the system transfer function evaluated at $s = 0$ (why?)

Time response of first-order systems

- First-order systems $G(s) = \frac{C(s)}{R(s)} = \frac{K}{\tau s + 1}$
- Step response
- System dc gain
- Ramp response

$$R(s) = 1/s^2,$$

$$C(s) = \frac{1}{s^2}\frac{K/\tau}{s + 1/\tau} = \frac{K}{s^2} - \frac{K\tau}{s} + \frac{K\tau}{s + 1/\tau},$$

$$c(t) = Kt - K\tau + K\tau\, e^{-t/\tau}, \quad t > 0$$

Steady-state response

$$c_{ss}(t) = Kt - K\tau$$

Time response of second-order systems

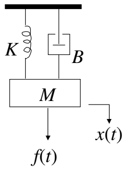
- Second-order systems

$$G(s)=\frac{C(s)}{R(s)}=\frac{b_0}{s^2+a_1 s+a_0}=\frac{\omega_n^{\ 2}}{s^2+2\varsigma\omega_n s+\omega_n^{\ 2}}$$

Natural frequency

Damping ratio

  - An example:



$$M\frac{d^2x}{dt^2}=f(t)-B\frac{dx}{dt}-Kx$$

$$G(s)=\frac{X(s)}{F(s)}=\frac{1}{Ms^2+Bs+K}$$

Time response of second-order systems

- Second-order systems $G(s)=\frac{\omega_n^{\ 2}}{s^2+2\varsigma\omega_n s+\omega_n^{\ 2}}$
- Step response
  - Case 1: $\zeta<1$ (underdamped), including $\zeta=0$ (undamped)

$$\boxed{c(t)=1-\frac{1}{\beta}e^{-\varsigma\omega_n t}\sin(\beta\omega_n t+\theta),}$$ where $\beta=\sqrt{1-\varsigma^2}$

and $\theta=\tan^{-1}(\beta/\varsigma)$

  - Case 2: $\zeta>1$ (overdamped)

$$\boxed{c(t)=1+k_1e^{-t/\tau_1}+k_2e^{-t/\tau_2},}$$

where $\tau_{1,2}=1/\left(\varsigma\omega_n\pm\omega_n\sqrt{\varsigma^2-1}\right)$

  - Case 3: $\zeta=1$ (critically damped)

$$\boxed{c(t)=1+k_1e^{-t/\tau}+k_2te^{-t/\tau},}$$ where $\tau=1/\omega_n$

Time response of second-order systems

- Second-order systems $G(s)=\frac{\omega_n^{\ 2}}{s^2+2\varsigma\omega_n s+\omega_n^{\ 2}}$
- Step response

Case 1: $\zeta<1$ (underdamped)

$$\boxed{c(t)=1-\frac{1}{\beta}e^{-\varsigma\omega_n t}\sin(\beta\omega_n t+\theta),}$$

Case 2: $\zeta>1$ (overdamped)

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Case 3: $\zeta=1$ (critically damped)

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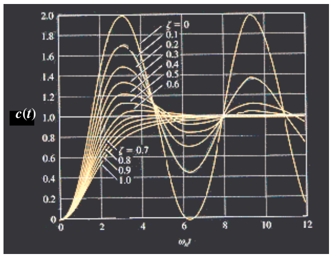


Examples of underdamped, overdamped, and critically damped responses:

(1) An animation (refer to the mov file)

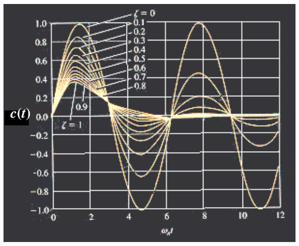
(2) Heavy public doors with dashpots

(3) An example of sound

Time response of second-order systems

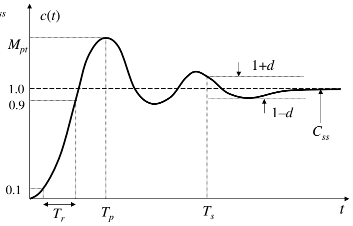
- Second-order systems $G(s)=\frac{\omega_n^{\ 2}}{s^2+2\varsigma\omega_n s+\omega_n^{\ 2}}$
- Step response
  - Case 1
  - Case 2
  - Case 3
  - Initial condition and impulse response

The initial condition excitation of higher-order systems cannot be modeled as simply as that of the first-order system; however, the impulse response of any system does give an indication of the nature of the initial-condition response, and thus the transient response



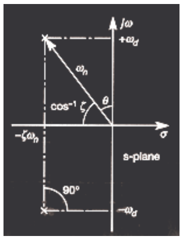
Time response of specifications in design

- Some parameters
  - Rise time, T_r
  - Peak value of the step response, M_{pt} ; time to reach it, T_p (how to calculate T_p?)
  - Steady state value, C_{ss}
  - Percent overshoot, $\frac{M_{pt}-C_{ss}}{C_{ss}}\times 100$
  - Settling time, T_s (how to calculate T_s?)



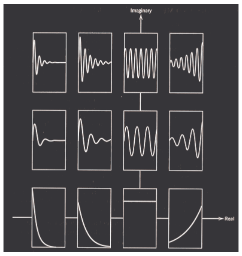
Time response of specifications in design

- Some parameters
- Time response and pole locations
  - The settling time is inversely related to the real part of the poles (the speed of response is increased by moving the poles to the left in the s-plane) $\boxed{T_s = k\tau = \frac{k}{\varsigma\omega_n}}$
  - Decreasing the angle $\cos^{-1}\zeta$ (increasing ζ) reduces the percent overshoot



Time response of specifications in design

- Some parameters
- Time response and pole locations
  - The settling time is inversely related to the real part of the poles (the speed of response is increased by moving the poles to the left in the s-plane)
  - Decreasing the angle $\cos^{-1}\zeta$ (increasing ζ) reduces the percent overshoot



This picture shows how changing pole locations in the *s*-plane affects responses

Frequency response of systems

- Frequency response: steady-state response of systems to sinusoidal inputs

$$r(t) = A\cos\omega_1 t, \quad R(s) = \frac{As}{s^2+\omega_1^2},$$

Assume that $\lim_{t\to\infty} c_g(t) = 0$

$$C(s) = G(s)R(s) = \frac{k_1}{s-j\omega_1} + \frac{k_2}{s+j\omega_1} + C_g(s)$$

$$k_1 = \frac{1}{2}AG(j\omega_1),\ k_2 = \frac{1}{2}AG(-j\omega_1), \quad \boxed{G(j\omega_1) = \left|G(j\omega_1)\right|e^{j\phi(\omega_1)}}$$

$$c_{ss}(t) = k_1 e^{j\omega_1 t} + k_2 e^{-j\omega_1 t} = A\left|G(j\omega_1)\right|\frac{e^{j(\omega_1 t+\phi(\omega_1))} + e^{-j(\omega_1 t+\phi(\omega_1))}}{2}$$

$$= A\left|G(j\omega_1)\right|\cos(\omega_1 t + \phi(\omega_1))$$

Frequency response of systems

- Frequency response: steady-state response of systems to sinusoidal inputs

$$r(t) = A\cos\omega_1 t, \quad G(j\omega_1) = \left|G(j\omega_1)\right|e^{j\phi(\omega_1)}$$

$$\boxed{c_{ss}(t) = A\left|G(j\omega_1)\right|\cos(\omega_1 t + \phi(\omega_1))}$$

  - The steady-state gain of a system for a sinusoidal input is the magnitude of the transfer function evaluation at $s = j\omega_1$, and the phase shift of the output sinusoid relative to the input sinusoid is the angle of $G(j\omega_1)$

Frequency response of systems

- Frequency response: steady-state response of systems to sinusoidal inputs
  - The steady-state gain of a system for a sinusoidal input is the *magnitude* of the transfer function evaluation at $s = j\omega_1$, and the *phase shift* of the output sinusoid relative to the input sinusoid is the angle of $G(j\omega_1)$
  - $G(j\omega)$ is defined as the frequency response function

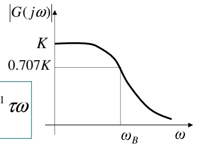
$$G(j\omega) = \left|G(j\omega)\right|e^{j\phi(\omega)}$$

Frequency response of systems

- Frequency response: steady-state response of systems to sinusoidal inputs
- Frequency response of first-order systems

$$G(s) = \frac{K}{\tau s + 1}$$

$$\boxed{\left|G(j\omega)\right| = \frac{K}{\left(1+\tau^2\omega^2\right)^{1/2}}, \quad \phi(\omega) = -\tan^{-1}\tau\omega}$$



  - System bandwidth, ω_B: The frequency at which the gain is equal to 1/sqrt(2) (approximately 0.707) times the gain at very low frequencies

Frequency response of systems

- Frequency response: steady-state response of systems to sinusoidal inputs
- Frequency response of first-order systems
- Frequency response of second-order systems

$$G(s)=\frac{\omega_n^2}{s^2+2\varsigma\omega_n s+\omega_n^2}=\frac{1}{(s/\omega_n)^2+2\varsigma(s/\omega_n)+1}$$

$$G(j\omega)=\frac{1}{[1-(\omega/\omega_n)^2]+j2\varsigma(\omega/\omega_n)}$$

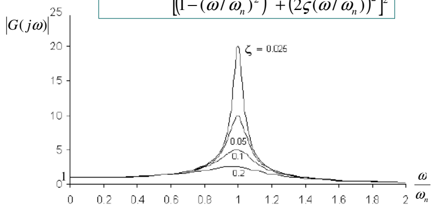
$$\boxed{\left|G(j\omega)\right|=\frac{1}{\left[\left(1-(\omega/\omega_n)^2\right)^2+\left(2\varsigma(\omega/\omega_n)\right)^2\right]^{\frac{1}{2}}}}$$

Question: What will happen if $\zeta=0$ and $\omega=\omega_n$?

Frequency response of systems

- Frequency response: steady-state response of systems to sinusoidal inputs
- Frequency response of first-order systems
- Frequency response of second-order systems

$$\boxed{\left|G(j\omega)\right|=\frac{1}{\left[\left(1-(\omega/\omega_n)^2\right)^2+\left(2\varsigma(\omega/\omega_n)\right)^2\right]^{\frac{1}{2}}}}$$



References

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- http://www.ketchum.org/bridgecollapse.html
- http://www.scar.utoronto.ca/~pat/fun/movies1d.html