ECE 1673: Linear Control Systems (4 Credits, Spring 2009)

Lecture 2: Mathematical Foundation

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Lab 1 and Homework 1

- Lab 1
  - Available on the course webpage
  - Due 1/21 (Wednesday) in class (right before lecture starts)
- Homework 1
  - B.3 (a)-(d), B.6, 2.12, and 2.13 in the text book
  - Due 1/14 (Wednesday) in class (right before lecture starts)

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Review of last lecture

- Important concepts
  - Open-loop and closed-loop systems
    - Example: Clock and heart; fast movement and slow movement
  - Feedback
    - Advantages and disadvantages
  - Stability
    - Example: Controlling unstable aircraft

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Outline of this lecture

- Something interesting from BioE 2696/ECE 2695 Control Theory in Neuroscience
- Mathematical foundation
  - Complex variables
  - Differential equations
  - Laplace transform

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Complex variables

- Number system
  - Natural number
  - Integer
  - Rational number

    Question: Why $\sqrt{2}$ is not a rational number?
  - Real number
  - Complex number

    Question: How complex numbers can be applied to "the real world"?

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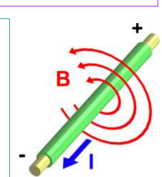
Complex variables

- Number system

Question: How complex numbers can be applied to "the real world"?

Examples of the application of complex numbers:
(1) Electric field and magnetic field.
(2) Complex numbers can be interpreted as being the combination of a phase and a magnitude, e.g., impedance in electric circuits.
(3) Complex numbers sometimes provide a quicker way to solve certain problems (it does appear that some mathematicians have absolutely no intuitive clue concerning the objects they are working with).



$$\int_0^T e^t \cos t \, dt = ?$$

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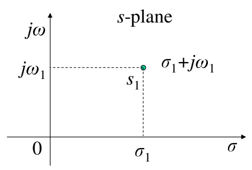
Complex variables

- Number system
- Complex variable
  - A complex variable s has two components: real component σ and imaginary component ω
  - Complex s-plane



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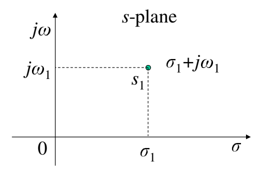
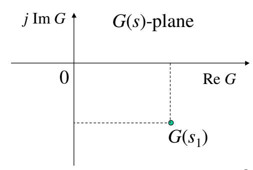
Complex variables

- Number system
- Complex variable
- Functions of a complex variable
  - Function $G(s) = \text{Re}\, G(s) + j\,\text{Im}\, G(s)$





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Complex variables

- Number system
- Complex variable
- Functions of a complex variable
- Analytic function
  - A function $G(s)$ of the complex variable s is called an analytic function in a region of the s-plane if the function and all its derivatives exist in the region
  - Example: $G(s) = \dfrac{1}{s(s+1)}$ is analytic at every point in the s-plane except at the points $s = 0$ and $s = -1$

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Complex variables

- Number system
- Complex variable
- Functions of a complex variable
- Analytic function
- Singularities and poles of a function
  - The singularities of a function are the points in the s-plane at which the function or its derivatives do not exist
  - Definition of a pole: if a function $G(s)$ is analytic in the neighborhood of s_i, it is said to have a pole of order r at $s = s_i$ if the limit

$$\lim_{s \to s_i} (s - s_i)^r G(s)$$

has a finite, nonzero value

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has a finite, nonzero value. In other words, the denominator of $G(s)$ must include the factor $(s - s_i)^r$, so when $s = s_i$, the function becomes infinite. If $r = 1$, the pole at $s = s_i$ is called a simple pole

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Complex variables

- Number system
- Complex variable
- Functions of a complex variable
- Analytic function
- Singularities and poles of a function
  - The singularities of a function are the points in the s-plane at which the function or its derivatives do not exist
  - Definition of a pole: if a function $G(s)$ is analytic in the neighborhood of s_i, it is said to have a pole of order r at $s = s_i$ if the limit $\lim_{s \to s_i} (s - s_i)^r G(s)$ has a finite, nonzero value. In other words, the denominator of $G(s)$ must include the factor $(s - s_i)^r$, so when $s = s_i$, the function becomes infinite. If $r = 1$, the pole at $s = s_i$ is called a simple pole.
  - Examples:

$$G(s) = \frac{10(s+2)}{s(s+1)(s+3)^3}; \qquad G(s) = e^{-5s}$$

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Complex variables

- Number system
- Complex variable
- Functions of a complex variable
- Analytic function
- Singularities and poles of a function
- Zeros of a function
  - Definition: If a function $G(s)$ is analytic at $s = s_i$, it is said to have a zero of order r at $s = s_i$ if the limit

$$\lim_{s \to s_i} (s - s_i)^{-r} G(s)$$

has a finite, nonzero value. Or, simply, $G(s)$ has a zero of order r at $s = s_i$ if $1/G(s)$ has an r-th order pole at $s = s_i$

Differential equations

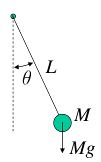
- Linear ordinary differential equations
  - A wide range of systems in engineering are modeled mathematically by differential equations
  - In general, the differential equation of an n-th order system is written

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = f(t)$$

Any examples?

Differential equations

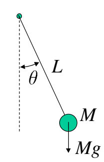
- Linear ordinary differential equations
- Nonlinear differential equations
  - Example



$$ML\frac{d^2\theta(t)}{dt^2} + Mg\sin\theta(t) = 0$$

Differential equations

- Linear ordinary differential equations
- Nonlinear differential equations
  - Example
  - Linearization of nonlinear differential equations



$$ML\frac{d^2\theta(t)}{dt^2} + Mg\sin\theta(t) = 0$$

For small value of θ

$$ML\frac{d^2\theta(t)}{dt^2} + Mg\theta(t) = 0$$

Differential equations

- Linear ordinary differential equations
- Nonlinear differential equations
- Solving linear differential equations with constant coefficients
  - Example:

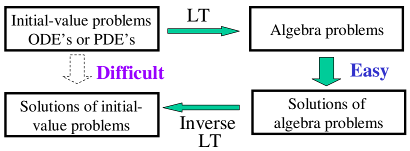
$$\frac{d^3 y}{dx^3} - \frac{dy}{dx} = 2x + 1 - 4\cos x + 2e^x$$

Differential equations

- Linear ordinary differential equations
- Nonlinear differential equations
- Solving linear differential equations with constant coefficients
  - Example: $\frac{d^3 y}{dx^3} - \frac{dy}{dx} = 2x + 1 - 4\cos x + 2e^x$
  - Classical method
    - To find the general homogeneous solution (involving solving the characteristic equation)
    - To find a particular solution of the complete nonhomogeneous equation (involving constructing the family of a function)
    - To solve the initial value problem

Differential equations

- Linear ordinary differential equations
- Nonlinear differential equations
- Solving linear differential equations with constant coefficients
  - Example
  - Classical method
  - Laplace transform



Differential equations

- Linear ordinary differential equations
- Nonlinear differential equations
- Solving linear differential equations with constant coefficients
  - Example
  - Classical method
  - Laplace transform

Examples (about "usefulness" of mathematical transforms)
(1) log and exp pairs.
(2) Multiplication of polynomials.
(3) Compact representation of data.

Laplace transform

- The Laplace transform of a function $f(t)$ is defined as

$$F(s) = L[f(t)] = \int_0^{\infty} f(t)e^{-st}dt$$

Laplace transform

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- The inverse Laplace transform is given by

$$f(t) = L^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}ds$$

Laplace transform

- The Laplace transform of a function $f(t)$ is defined as

$$F(s) = L[f(t)] = \int_0^{\infty} f(t)e^{-st}dt$$

- The inverse Laplace transform is given by

$$f(t) = L^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma - \infty}^{\sigma + \infty} F(s)e^{st}ds$$

  - We seldom use the above equation to calculate an inverse Laplace transform; instead we use the equation of Laplace transform to construct a table of transforms for useful time functions. Then we use the table to find the inverse transform

Laplace transform table

TIME DOMAIN		FREQUENCY DOMAIN
$\delta(t)$	unit impulse	1
A	step	$\frac{A}{s}$
t	ramp	$\frac{1}{s^2}$
t^2		$\frac{2}{s^3}$
$t^n, n > 0$		$\frac{n!}{s^{n+1}}$
e^{-at}	exponential decay	$\frac{1}{s+a}$
$\sin(\omega t)$		$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$		$\frac{s}{s^2+\omega^2}$
te^{-at}		$\frac{1}{(s+a)^2}$
t^2e^{-at}		$\frac{2!}{(s+a)^3}$

Laplace transform table (continued)

TIME DOMAIN	FREQUENCY DOMAIN
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\left[B\cos\omega t+\left(\frac{C-aB}{\omega}\right)\sin\omega t\right]$	$\frac{Bs+C}{(s+a)^2+\omega^2}$
$2\lvert A\rvert e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{s+\alpha-\beta j}+\frac{A^{\text{complex conjugate}}}{s+\alpha+\beta j}$
$2t\lvert A\rvert e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{(s+\alpha-\beta j)^2}+\frac{A^{\text{complex conjugate}}}{(s+\alpha+\beta j)^2}$
$\frac{(c-a)e^{-at}-(c-b)e^{-bt}}{b-a}$	$\frac{s+c}{(s+a)(s+b)}$
$\frac{e^{-at}-e^{-bt}}{b-a}$	$\frac{1}{(s+a)(s+b)}$

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Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function

$$F(s)=\frac{b_m s^m+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=\frac{N(s)}{D(s)},\quad m<n$$

– Example: $$\frac{c}{(s+a)(s+b)}=\frac{k_1}{s+a}+\frac{k_2}{s+b}$$

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Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function

$$F(s)=\frac{b_m s^m+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=\frac{N(s)}{D(s)},\quad m<n$$

– Case 1: $D(s)$ does not have repeated roots. Then $F(s)$ can be expressed as

$$F(s)=\frac{N(s)}{\prod_{i=1}^{n}(s-p_i)}=\frac{k_1}{s-p_1}+\cdots+\frac{k_n}{s-p_n},$$

where $k_j=(s-p_j)F(s)\big|_{s=p_j}$.

k_j is also called the residue of $F(s)$ in the pole at $s=p_j$

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Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function

$$F(s)=\frac{b_m s^m+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=\frac{N(s)}{D(s)},\quad m<n$$

– Case 1: $D(s)$ does not have repeated roots

– Case 2: $D(s)$ has repeated roots. Then $F(s)$ can be expanded as in the example

$$F(s)=\frac{N(s)}{(s-p_1)(s-p_2)^r}=\frac{k_1}{s-p_1}+\frac{k_{21}}{s-p_2}\cdots+\frac{k_{2r}}{(s-p_2)^r},$$

where $$k_{2j}=\frac{1}{(r-j)!}\frac{d^{r-j}}{ds^{r-j}}[(s-p_2)^r F(s)]\Big|_{s=p_2}$$

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Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function

$$F(s)=\frac{b_m s^m+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=\frac{N(s)}{D(s)},\quad m<n$$

– Case 1: $D(s)$ does not have repeated roots

– Case 2: $D(s)$ has repeated roots

– Examples: Find inverse Laplace transforms of the following functions

$$F_1(s)=\frac{5}{s^2+3s+2},\quad F_2(s)=\frac{2s+3}{s^3+2s^2+s}$$

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Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- Theorems of the Laplace transform
  - Final value theorem

$$\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$$

The final-value theorem is valid only if $sF(s)$ does not have any poles on the $j\omega$ axis and in the right half of the s-plane.

Examples:

$$F_1(s)=\frac{5}{s(s^2+s+2)},\quad F_2(s)=\frac{\omega}{s^2+\omega^2}$$

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Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- Theorems of the Laplace transform
  - Final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

The final-value theorem is valid only if $sF(s)$ does not have any poles on the $j\omega$ axis and in the right half of the s-plane.

Examples: $F_1(s) = \frac{5}{s(s^2+s+2)}$, $F_2(s) = \frac{\omega}{s^2+\omega^2}$

$\lim_{t\to\infty} f_1(t) = 5/2 \qquad f_2(t) = \sin\omega t$

Final value theorem does not apply in the second example.

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Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- Theorems of the Laplace transform
  - Final value theorem
  - Differential theorem

$$L\left[\frac{df}{dt}\right] = sF(s) - f(0^-),$$

$$L\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0^-) - \cdots - f^{n-1}(0^-),$$

$$\text{where } f(0^-) = \lim_{t\to 0} f(t), \quad t<0$$

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Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- Theorems of the Laplace transform
  - Final value theorem
  - Differential theorem

Note that in the text book

$$L\left[\frac{df}{dt}\right] = sF(s) - f(0^+),$$
$$\text{where } f(0^+) = \lim_{t\to 0} f(t), \quad t>0$$

$$L\left[\frac{df}{dt}\right] = sF(s) - f(0^-),$$

$$L\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0^-) - \cdots - f^{n-1}(0^-),$$

$$\text{where } f(0^-) = \lim_{t\to 0} f(t), \quad t<0$$

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Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- Theorems of the Laplace transform
  - Final value theorem
  - Differential theorem
  - Integral theorem

$$L\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$$

  - Shifting theorem

$$L[f(t-t_0)u(t-t_0)] = e^{-t_0 s}F(s)$$

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Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- Theorems of the Laplace transform
  - Final value theorem
  - Differential theorem
  - Integral theorem
  - Shifting theorem
  - Frequency shift theorem

$$L[e^{-at}f(t)] = F(s+a)$$

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Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- Theorems of the Laplace transform
  - Final value theorem
  - Differential theorem
  - Integral theorem
  - Shifting theorem
  - Frequency shift theorem

$$L[f(t-t_0)u(t-t_0)] = e^{-t_0 s}F(s)$$

Same signs

$$L[e^{-at}f(t)] = F(s+a)$$

Different signs

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Laplace transform

- The Laplace transform
- The inverse Laplace transform
- Partial fraction expansion of a rational function
- Theorems of the Laplace transform
  - Final value theorem
  - Differential theorem
  - Integral theorem
  - Shifting theorem
  - Frequency shift theorem
  - Theorem of convolution integral

$$L^{-1}[F_1(s)F_2(s)] = \int_0^t f_1(t-\tau)f_2(\tau)d\tau = \int_0^t f_1(\tau)f_2(t-\tau)d\tau$$

"For the modern EE, the lure of the Laplace transform is its ability to map the complicated operation of convolution into multiplication. This integral has for decades driven electrical engineering undergraduates to contemplate theology either for salvation or as an alternative career."

— Paul J. Nahin

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