NOTES for Fiber Optics

This lecture covers Chapter 9

1. Optical fiber, merdional rays, skewed rays, numerical aperture
2. Step index fiber, V-number,
3. Weakly guiding fiber, LP modes,
4. Cutoff mode, mode numbers, propagation constants, group velocities

Note:

- The derivation in this note was based on handout to be distributed in class (Make sure attend class). Essentially, we cover Section 2.3 in a better detail.

Optical Fibers

Optical fiber is a waveguide with two-dimensional confinement. In 1-D waveguide, we have wave vector only confined in one direction (y-direction), for waveguide

$$\vec{k} = \vec{k}_y + \vec{k}_z$$

Along y-direction, light is confined in an optical resonator, which limit the number of the mode (i.e. not all the angle smaller than the critical angle $\theta < \overline{\theta}_c$ can guide the light). Now in optical fiber, we have 2-D confinement, the wave vector has three components:

$$\vec{k} = \vec{k}_z + \vec{k}_r + \vec{k}_\phi \equiv \vec{\beta} + \vec{k}_r + \vec{k}_\phi$$

β: propagation constant

$\vec{k}_r, \vec{k}_\theta$: confined by fiber that will limit the # of propagation modes

Showing below, if

$\vec{k}_\phi = 0$: Meridional rays

$\vec{k}_\phi \neq 0$: Skewed rays

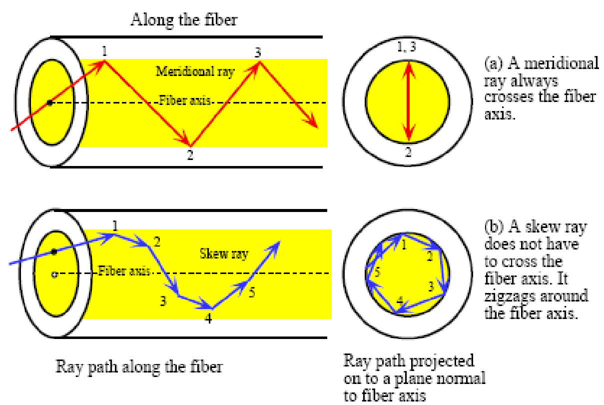


Illustration of the difference between a meridional ray and a skew ray. Numbers represent reflections of the ray.

Numerical Apertures:

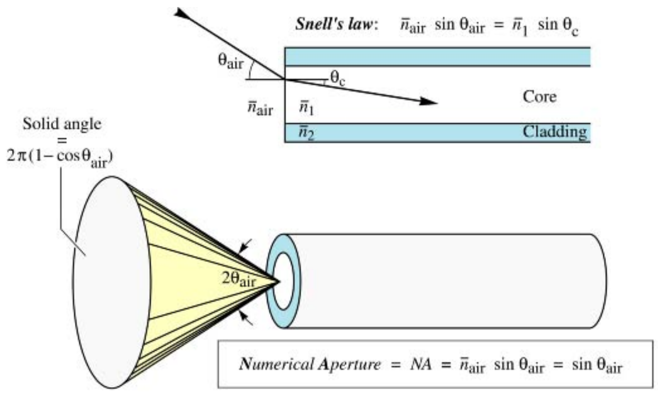


Fig. 12.6. Illustration of the *Numerical Aperture* (*NA*) of a fiber. For example, the light acceptance angle in air is $\theta_{air} = 11.5°$ for a numerical aperture $NA = 0.2$.

If we want to couple light in fiber and keep the light in, we need a coupling angle that make the incident light has an angle smaller than the critical angle $\bar{\theta}_c$, ($\theta_c = 90^o - \bar{\theta}_c$). What is the maximum angle allowed?

Well, the critical angle on the core-cladding interface is

$$\theta_c = 90^o - \bar{\theta}_c$$

$$\sin\theta_c = \frac{n_2}{n_1} = \sin\left(90^o - \bar{\theta}_c\right) = \cos\bar{\theta}_c$$

To couple light from free space (air) to the fiber, we have the maximum incident angle θ_a must satisfy **numerical aperture** of the fiber:

$$N.A. = 1\cdot\sin\theta_a = n_1\sin\bar{\theta}_c = n_1\sqrt{1-\left(\frac{n_2}{n_1}\right)^2} = \sqrt{n_1^2 - n_2^2}$$

Note: for the fundemental mode, the mode angle is the smallest among all the allowed mode angles. (i.e. $\theta_0 < \theta_m$). **For standard fiber NA=0.13**

How to find guide mode profile and transcendental equation for the propagation constants? We need to solve the Maxwell's EM equation. For each **E, B** component of the guided wave, EM equation must be satisfied:

$$\nabla^2 U + n^2 k_0^2 U = 0$$

In cylindrical coordinate, the EM equation becomes:

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} + n^2 k_0^2 U = 0$$

We know that the guided wave is propagating along z-axis, we have symmetry for ϕ that is $\phi = \phi + 2\pi$. We therefore can write the solution of this EM equation as:

$$U(r,\phi,z) = u(r)e^{-jl\phi}e^{-j\beta z}$$

Before get into math, **<u>what the mode is going to look like along ϕ???, for *l*=0, and *l*≠0???.</u>**

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left(n^2 k_0^2 - \beta^2 - \frac{l^2}{r^2}\right)u = 0$$

We know the propagation speed of the guided wave is between c/n_2 and c/n_1. So we have

$$n_2 k_0 < \beta < n_1 k_0$$

For the fiber core and cladding, we define:

$$k_T^2 = n_1^2 k_0^2 - \beta^2$$
$$\gamma^2 = \beta^2 - n_2^2 k_0^2$$
$$k_T^2 + \gamma^2 = \left(n_1^2 - n_2^2\right)k_0^2 = NA \cdot k_0^2$$

Plug in the EM wave equations above:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left(k_T^2 - \frac{l^2}{r^2}\right)u = 0 \quad r < a$$
$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \left(\gamma^2 + \frac{l^2}{r^2}\right)u = 0 \quad r > a$$

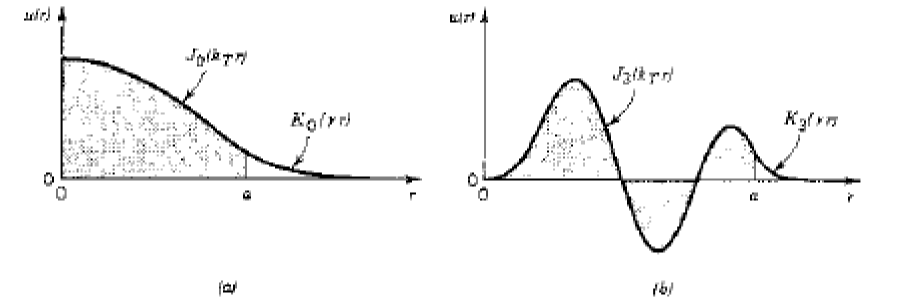
Solution will be

$$u(r) = \begin{cases} A \cdot J_l(k_T r) & r < a \\ B \cdot K_l(\gamma r) & r > a \end{cases}$$

$J_l(x)$ is the Bessel function of first kind (like a decayed sine function *sin(x)/x*)
$K_l(x)$ is the modified Bessel function of second kind (like ~ e^x/x)

Indeed, if x>>1, we will have approximation:

$$J_l(x) = \sqrt{\frac{2}{\pi x}}\cos\left[x - \left(l + \frac{1}{2}\right)\frac{\pi}{2}\right] \quad x << 1$$
$$J_l(x) = \sqrt{\frac{\pi}{2x}}\left(1 + \frac{4l^2 - 1}{8x}\right)e^{-x} \qquad x >> 1$$



Weak guided approximation (LP modes): How do you find the mode profile (i.e. determined propagation constant β, A, and B). The mathematical derivation is slightly difficult. However, if we have weak guiding (that are the case for most of optical fibers):

$$n_1 - n_2 << 1$$

We can approximate all the guided modes as TEM waves. For TEM wave, E and B fields only have transverse component,that is perpendicular to the propagation direction of z. In these cases, the boundary condition between the core and cladding become rather simple, we need E field and its derivative are continuous:

$$AJ_l(k_T a) = BK_l(\gamma a) \qquad (1)$$
$$Ak_T J_l^{'}(k_T a) = B\gamma K'_l(\gamma_T a) \quad (2)$$

Using (1)/(2), we have transcendental equation:

$$\frac{k_T a J_l^{'}(k_T a)}{J_l(k_T a)} = \frac{\gamma a K'_l(\gamma_T a)}{K_l(\gamma a)}$$

Note that if we set:

$$X = K_T a \quad Y = \gamma a \qquad K_T^2 + \gamma^2 = NA^2 \cdot k_0^2$$

We will have a relationship between *X* and *Y*:

$$X = K_T a \quad Y = \gamma a \qquad X^2 + Y^2 = \left(2\pi \frac{a}{\lambda_0} NA\right)^2 \equiv V^2$$

The transcendental equation can be re-written as

$$\frac{XJ_l^{'}(X)}{J_l(X)} = \frac{YK'_l(Y)}{K_l(Y)} \qquad X^2 + Y^2 = V^2$$

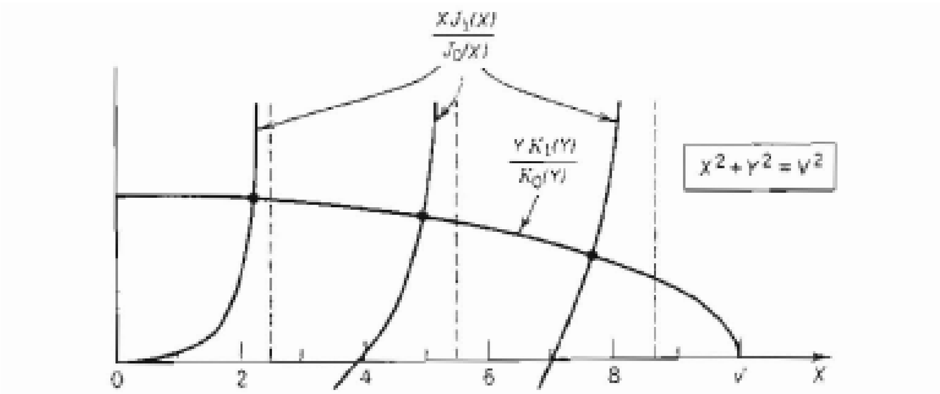
The reason we write in X, and Y form is become the solution of transcendental equation does NOT depend on the fiber structures and operational wavelength (a, λ). The solution is generalized, and we can refer to each individual fiber structure by introducing V-number

$$V = 2\pi \frac{a}{\lambda_0} NA$$

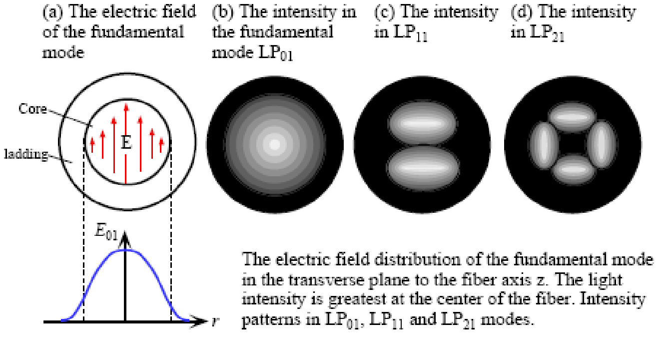
After applying a few mathematical tricks (see your handout). We have transcendental equation as:

$$\frac{XJ_{l\pm1}(X)}{J_l(X)} = \pm\frac{YK_{l\pm1}(Y)}{K_l(Y)} \qquad X^2 + Y^2 = V^2$$

This equation can be solved numerically and graphically shown below



For each *l=0,1,2…*, we apply the same graphical method to find solution x_{lm}, we will find all the propagation constant for all the mode.



Single mode condition:

If we take a close look of the above graph, we find that for each root of LHS ($J_{L\pm1}(x) = 0$), we will have a solution for the transcendental equation. However, for the variable X, it cannot be larger than V. So the solution for the transcendental equation cannot be larger than V!!!

$$X < V$$

$$X = V \quad Y = 0$$

We know that the smallest root for $J_0(x) = 0$, is $x_{01}=0$ (*l=0, m=1*). So fiber has least have one mode, the next smallest root is for $J_1(x) = 0$ $x_{11}=2.405$ (*l=1, m=1*). So, if V<2.405, we will only have one mode in fiber, we call this fiber single-mode fiber. **Single mode condition:**

$$V = 2\pi \frac{a}{\lambda_0} NA < 2.405$$

Multi-mode fibers:

If V number is large, we will have a lot of modes, the first thing we need to find out for a multimode fiber is how many modes do we have?

- **Number of the mode**

We know that for 0<X<V, each root of $J_{L\pm1}(x)=0$ will lead to a solution of the transcendental equation, and lead to a mode for the fiber. For each given *l=0,1,2…*, and X<V, how many root we can have for $J_{L\pm1}(x)$?? For V>>1, we have

$$J_{l\pm1}(x)=\sqrt{\frac{2}{\pi x}}\cos\left[x-\left(l\pm1+\frac{1}{2}\right)\frac{\pi}{2}\right]=0 \quad \Rightarrow \quad x_{lm}=\left(2m-1+l\pm1+\frac{1}{2}\right)\frac{\pi}{2}\approx(2m+l)\frac{\pi}{2}$$

For any given *l=0,1,2….*, The number of roots that $x_{lm}<V$ are

$$M_l \approx \frac{V}{\pi}-\frac{l}{2}$$

For *l*=0, we have the most number of Root that is

$$M_0 \approx \frac{V}{\pi}$$

For $l=\frac{2V}{\pi}$, we have the least number of Root that is

$$M_{l\max}=0$$

The total # of modes will be the area of the follow of the triangle:

$$M \approx \frac{4}{\pi^2}V^2$$

- **Propagation constants (V>>1)**

We can solve transcendental equation to find all the roots and therefore propagation constant. However, the root of transcendental equations can be approximated by the solution for (why??)

$$J_{l\pm1}(X)=0$$

When V is large, the solution of the above equation can be approximated as:

$$x_{lm}\approx(2m+l)\frac{\pi}{2}$$

So the propagation constant are

$$\beta_{lm}=\sqrt{n_1^2k_0^2-\frac{x_{lm}^2}{a^2}}=\sqrt{n_1^2k_0^2-(l+2m)^2\frac{\pi^2}{4a^2}}$$

We can further simply the above equation to (your assignment):

$$\beta_{lm}\approx n_1k_0\left[1-\frac{(l+2m)^2}{M}\Delta\right]$$

- **Group velocity (V>>1)**

Since we know the relationship between the propagation constant β_{lm} and ω, we can easily calculate the group velocity…

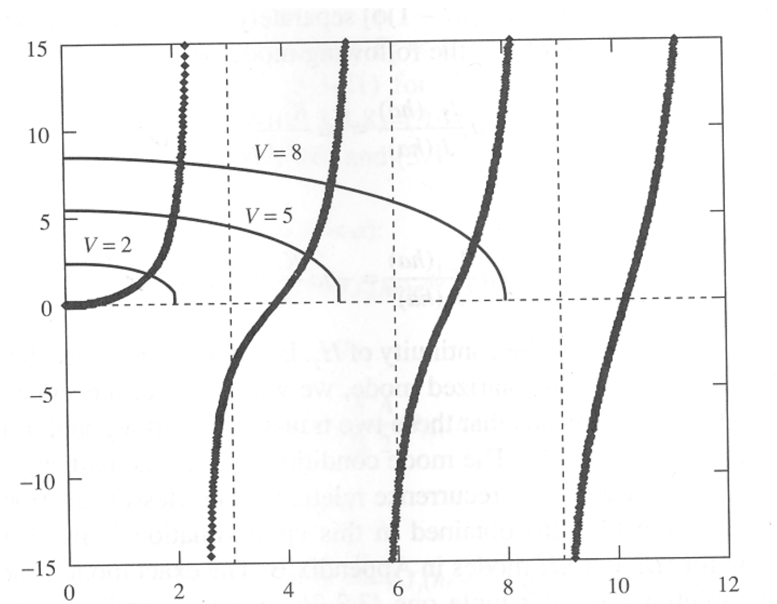
$$\beta_{lm}\approx n_1\frac{\omega}{c}\left[1-\frac{(l+2m)^2}{M}\Delta\right]$$

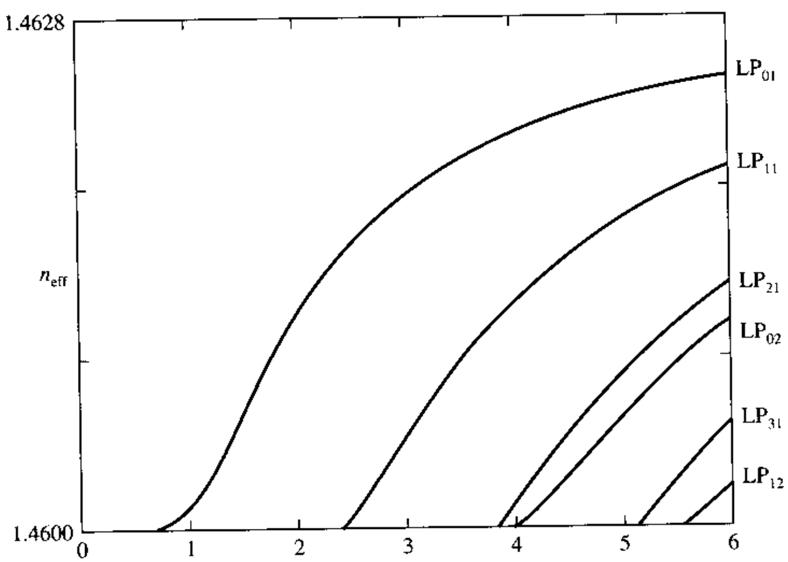
We have

$$\frac{d\beta_{lm}}{d\omega}\approx\frac{n_1}{c}\left[1-\frac{(l+2m)^2}{M}\Delta\right]$$

So the group velocity $\frac{d\omega}{d\beta_{lm}}$ becomes

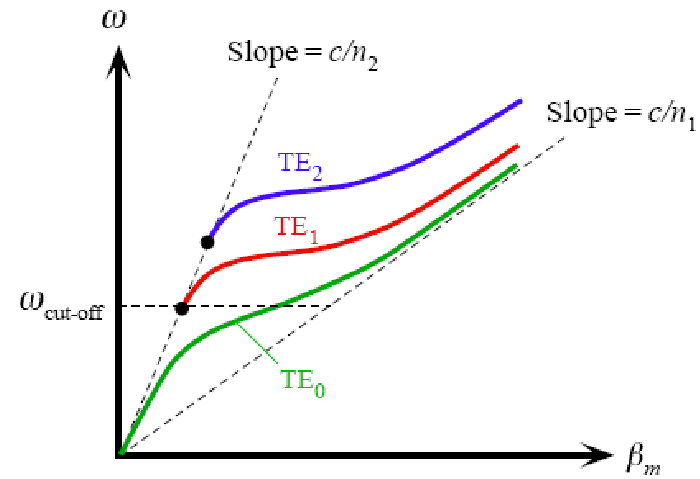
$$\frac{d\omega}{d\beta_{lm}}\approx\frac{c}{n_1}\left[1+\frac{(l+2m)^2}{M}\Delta\right]$$





Intermodal dispersion

In multimode operation, the lowest mode (m=0) has the slowest group velocity close to c/n_1, while higher mode propagate slower. This is become a good portion of the light is carried by the cladding where the refractive index is smaller.



Remember that dispersion is defined as "different part of "one optical pulse" propagate at different speeds that lead to a pulse broadening. The source of the different speed could come from an optical pulse containing different wavelength, or an optical pulse contains different propagation modes.

Now, what is the bulk part of this dispersion? What is the maximum propagation difference in a multimode fiber?

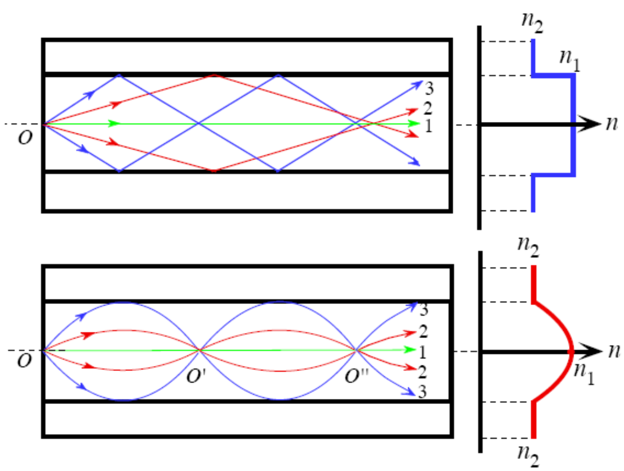
$$\Delta\tau = \frac{L}{c/n_1} - \frac{L}{c/n_2} = \frac{n_1 - n_2}{c}$$

So, the intermodal dispersion can be estimated as

$$D_m = \left(\frac{n_1}{c} - \frac{n_2}{c}\right) \equiv \Delta\left(\frac{1}{v}\right)$$

Graded Index (GRIN) fibers for intermodal dispersion reduction

How do you reduce the intermodal dispersion? Let's first take a look of a regular step index fiber



In the last lecture, we know that for a step index fiber, when mode number is large, we have

$$v_g = \frac{d\omega}{d\beta_{lm}} \approx \frac{c}{n_1}\left[1 - \frac{(l+2m)^2}{M}\Delta\right]$$

and

$$M \approx \frac{4}{\pi^2}V^2 \qquad \Delta = \frac{n_1 - n_2}{n_1}$$

So the maximum group velocity different will be when l+$2m$=Maximum, which is $\sqrt{M}$. The group velocity different will be $c/n_1\,\Delta$, that is depends on the index contrast.

Question: how good is our ballpark estimation?

We can reduce the intermodal group velocity difference using a graded index fiber. Note that the group velocity in a multimode fiber (with a lot of modes) reduces with l and m. (This is different from waveguide, and fiber with less number of modes). **Why?**

In this case, we interpret this by the figure shown in the previous page. If all the modes have similar penetration depth into cladding (similar evanescent wave), the mode speeds only depends on the mode angles. Higher order mode have larger angle, of course it propagates slower than the lower order mode. How do we solve this problem? How much benefit we will have???

We normally choose a index profile follow the power law.

$$n^2(r) = n_1^2\left[1 - 2\left(\frac{r}{a}\right)^p \Delta\right] \qquad r \le a$$

After a tedious mathematic derivation, we can reach that the group velocity for this profile is

$$v_q = \frac{c}{n_1}\left[1 - \frac{p-2}{p+2}\left(\frac{q}{M}\right)^{p/(p+2)}\Delta - \frac{4p-4}{p+2}\left(\frac{q}{M}\right)^{2p/(p+2)}\frac{\Delta^2}{2}\right]$$

Where $q = (l+2m)^2$, from 1 to M. so, group velocity difference will be (for p=2)

$$v_{\max} = \frac{c}{n_1} \qquad v_{\min} = \frac{c}{n_1}\left(1 - \frac{\Delta^2}{2}\right)$$

So, instead of step index fiber with group velocity difference of $c/n_1\,\Delta$, we now have index difference of $c/n_1\,\Delta^2/2$

Material dispersion

If the index of refraction is function of wavelength, an optical pulse travels in a dispersive medium of refractive index n with a group velocity v=c/N, where

$$N = n - \lambda_0 \frac{dn}{d\lambda}$$

Since the pulse is a wave packet, composed of a spectrum of components of different wavelengths each travel at different group velocity. The temporal width of the pulse will be widened

$$\Delta\tau = |D_\lambda|\sigma_\lambda L$$

$$D_\lambda = -\frac{\lambda}{c}\cdot\frac{d^2 n}{d\lambda^2}$$

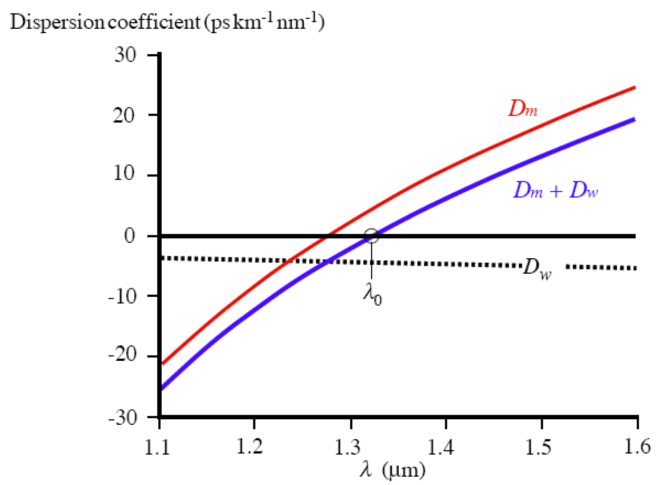
Waveguide dispersion

For a single-mode fiber, even the index of refraction of n_1 and n_2 are constant, dispersion still happens. Since the pulse is a wave packet, composed of a spectrum of components of different wavelengths.

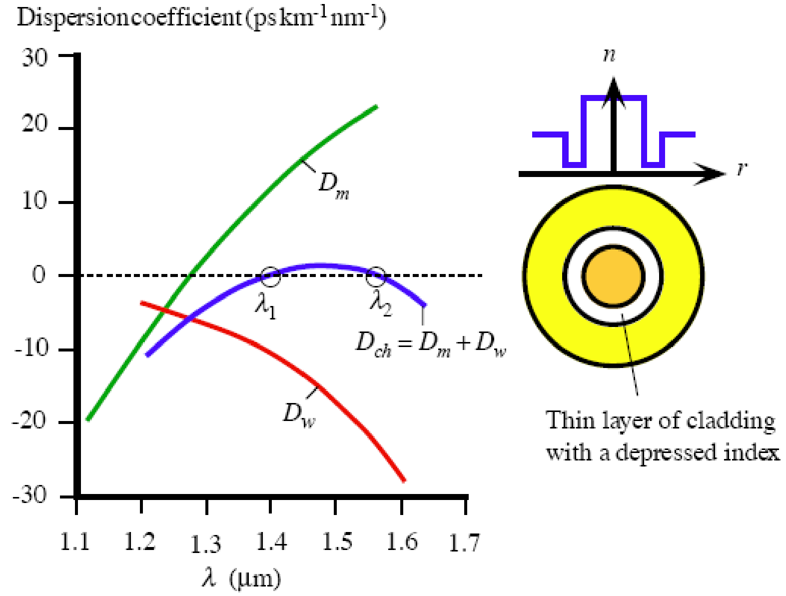
For each wavelength, we have different mode profile. Generally speaking, the guide mode profile expands with wavelength, if a wavepacket contains multiwavelength, different wavelength will have different mode index, which mean they travels at different velocities. This will lead to the dispersion. We refer this dispersion as **waveguide dispersion**. This can be related to the V-number by

$$\frac{1}{v_g} = \frac{d\beta}{d\omega} = \frac{d\beta}{dV}\cdot\frac{dV}{d\omega} = \frac{a\cdot\mathbf{NA}}{c}\cdot\frac{d\beta}{dV}$$

$$D_w = -\left(\frac{1}{2\pi c}\right)V^2\frac{d^2\beta}{dV^2}$$



We also can control the waveguide geometry to control the waveguide dispersion, that yield a total chromatic dispersion that is flattened between two wavelength λ_1 and λ_2. We refer this fiber as **dispersion flatten fiber.**



Bit rate, dispersion, bandwidth

For this part, please refer to section 2.6 of your text book in detail.

If we have very short optical pulses (binary information "1") (T~0), after a certain distance of propagation, we have a pulse with broader duration. If the full width at half power (**FWHP**) is $\Delta\tau_{1/2}$, the maximum **return-to-zero** (**RZ**) data rate B will be

$$B = \frac{1}{2\Delta\tau_{1/2}}$$

If our system can distinguish two binary 1s when they cross at FWHM, we will double the data rate B, which is **nonreturen-to-zero (NRZ)** data rate

$$B = \frac{1}{\Delta\tau_{1/2}}$$

If we have a Gaussian pulse

$$h(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}$$

Where we have

$$\sigma = 0.425\Delta\tau_{1/2}$$

where we refer σ as root-mean-square (rms) dispersion or standard deviation from the mean. If we require the adjacent pulse must have minimum 4σ separations, the RZ data rate will be

$$B = \frac{1}{4\sigma} = \frac{0.59}{\Delta\tau_{1/2}}$$

For a Gaussian pulse, if the rms spectrum width is σ_λ, the rms pulse width after propagating down to a fiber with distance *L* becomes:

$$\sigma = LD_{ch}\sigma_\lambda$$

Where D_{ch} is the chromatic dispersion coefficient, the product of the data rate and propagation distance becomes

$$BL = \frac{0.25L}{\sigma} = \frac{0.25}{D_{ch}\sigma_\lambda}$$